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OF THE

EDINBURGH MATHEMATICAL SOCIETY.

VOLUME XXI.

SESSION 1902-1903.

WILLIAMS AND NORGATE,
14 HENRIETTA STREET, COVENT GARDEN, LONDON; AND
20 SOUTH FREDERICK STREET, EDINBURGH
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[redacted]

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INDEX.

ERRATUM.

Page 136, line 16, for A_2B_2 and A_3B_3 read A_1B_2 and A_1B_3 respectively.

PROCEEDINGS

OF THE

EDINBURGH MATHEMATICAL SOCIETY.

TWENTY-FIRST SESSION, 1902-1903.

First Meeting, 14*th November*, 1902.

Dr THIRD in the Chair.

For this Session the following Office-bearers were elected:—

President - - - Mr JOHN A. THIRD, M.A., D.Sc.

Vice-President - Mr CHARLES TWEEDIE, M.A., B.Sc., F.R.S.E.

Hon. Secretary - Mr DONALD C. M'INTOSH, M.A., F.R.S.E.

Hon. Treasurer - Mr JAMES ARCHIBALD, M.A.

Editors of Proceedings.

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Mr W. L. THOMSON, M.A., B.A.

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Mr JAMES GOODWILLIE, M.A., B.Sc., F.R.S.E.

Mr JOHN TURNER, M.A., B.Sc.

Mr ALEX. G. BURGESS, M.A., F.R.S.E.

Mr J. A. MACDONALD, M.A., B.Sc.

[*The following Correspondence was submitted at the Second Meeting (12th December* 1902).]

Mathematical Correspondence

ROBERT SIMSON, MATTHEW STEWART, JAMES STIRLING.

[The correspondence which is here printed was bought by me on the 28th of March 1887 at the sale of the Gibson-Craig collection of Scottish MSS.

Simson's letters, which are beautifully written, seem all to have passed through the post, but Stewart's letters are, I conjecture, merely the drafts of what he proposed to send. The handwriting of the latter, though legible, is not elegant, and there are frequent erasures. I have scrupulously respected, in all the letters, the spelling, the punctuation (or want of it), the use or disuse of capitals, and I have made no attempt to improve the style.

A few of Simson's letters are given in the "Account of the Life and Writings of Robert Simson" by the Rev. William Trail (1812), and a few more in some articles, "Geometry and Geometers," by Thomas Stephens Davies in the *Philosophical Magazine*, 3rd series, vol. 33, pp. 201-206, 513–524 (1848).

In one of these articles [*Philosophical Magazine*, 3rd series, vol. 37, p. 198 (1850)] Davies states that he applied to an eminent archæologist to ascertain what had become of Matthew Stewart's MSS. "In a short time he sent me the copy of a letter from the proper custodian of the papers, decisive on this head. *They are all destroyed—deliberately burnt;* and not only his, *but likewise all the MS. of his son Dugald Stewart.*"

The following remarks and dates regarding the authors of the correspondence may save the reader the trouble of consulting a cyclopædia or dictionary of biography; if they do not, so much the better.

Robert Simson was born in Ayrshire on 14th October 1687. In 1711 he was appointed professor of mathematics in Glasgow University, where he had studied, and one of the first subjects

which attracted his attention was Euclid's Porisms. Simson was the first of the modern mathematicians who discovered, from the description left by Pappus, what the nature of the ancient porisms was. He published a paper on this subject in the *Philosophical Transactions* of the Royal Society of London in 1723, but his treatise *De Porismatibus* did not appear till after his death. His *Sectiones Conicae* was issued in 1735; his restoration of Apollonius's *Loci Plani* was finished about 1738 but not published till 1749. His *Elements of Euclid* both in Latin and English appeared in 1756, and in 1762 he added Euclid's book of *Data*. In 1761 he resigned his professorship, and on 1st October 1768 he died. In 1776 his *Opera Quaedam Reliqua* was edited by James Clow, to whom all Simson's manuscripts were bequeathed, and printed at the expense of Philip, Earl Stanhope. This work contains a restoration of Apollonius's treatise *De Sectione Determinata* with the addition of two more books, the *Porismatum Liber*, a book on *Logarithms*, and a fragment on the *Limits of Quantities and Ratios*.

Matthew Stewart was born at Rothesay in the island of Bute in 1717. He entered the University of Glasgow in 1734, and in 1741 went to the University of Edinburgh to prepare for entering the Church. He attended the lectures of Maclaurin during session 1742–3, on 6th May 1744 was licensed by the Presbytery of Dunoon, and on 9th May 1745 was presented by the Duke of Argyll to the living of Roseneath. For the chair rendered vacant by the death of Maclaurin in June 1746, Stewart became a candidate, and issued his *Some General Theorems of considerable use in the higher parts of mathematics* (the preface is dated October 1, 1746). Stewart obtained the chair in September 1747, and discharged the duties of it till 1772, when his health gave way. His distinguished son Dugald undertook to lecture in his stead, and in 1775 was appointed joint-professor. Stewart died on 23rd January 1785.

In 1756 Stewart gave, in the second volume of the Essays of the Philosophical Society of Edinburgh, a solution of Kepler's problem; in 1761 he published *Tracts, Physical and Mathematical*; and in 1763 *Propositiones Geometricae more Veterum demonstratae*. An obituary notice of him by Professor John Playfair will be found in the 1st volume of the *Transactions* of the Royal Society of Edinburgh.

James Stirling was born at Garden in Stirlingshire in 1692. He was educated at Glasgow University, and matriculated at Oxford on 18th January 1710-11. In 1715 he was expelled from Oxford for corresponding with noted Jacobites, and went to Venice. While he was there he made the acquaintance of Niclaus Bernoulli, who was professor of mathematics at Padua. He returned to London about 1725, and in the following year was elected a Fellow of the Royal Society. In 1735 he was appointed manager to the Scots Mining Company at Leadhills. But for his Jacobite principles he might have been Maclaurin's successor. He died in Edinburgh on 5th December 1770. He published in 1717 a commentary on Newton's lines of the third order, *Lineae tertii ordinis Neutonianae, sive illustratio tractatus D. Neutoni de enumeratione linearum tertii ordinis*, and added two new kinds to the seventy-two which Newton had remarked among curves of the third order. At p. 32 of this work occurs the theorem which usually goes by the name of Maclaurin's theorem. In 1730 he published his *Methodus differentialis seu de summatione et interpolatione serierum infinitarum*.

J. S. MACKAY]

To
MR ROBERT SIMSON
Professor of Mathematics
in the University of Glasgow

EDR. *Jan.* 3, 1741

SIR

I expected to have had the Pleasure of seeing you at Glasgow on my way here, but as you was otherwise taken up so that I could not see you then, I hope Sir, you will excuse my giving you the trouble of this Letter. In October last I found out a Theorem which I thought might be of some service towards finding the Quadrature of the Hyperbola and very soon afterwards I found a Construction of that Problem. I delayd doing any further till I should have occasion to see you at Glasgow and know from you what Gregory in his Vera Circuli et Hyperbolæ Quadratura had done towards the solving of this Problem, I not having access to see that Book in Bute, but being disapointed of seeing you at Glasgow I expected when I came here to have access to see that Book, and

accordingly I enquird for it at the Publick Library but was told that Mr M'Laurin had it from the Library, so that I do not expect to see that Book I not having occasion to be the least acquainted with Mr M'Laurin. I have therefore Sir, usd the Freedom with you, to send you the Theorem and the Construction and humbly Intreat the Favour of you to let me know your opinion of them and what Gregory has done, if you will be so good as to write me within a post or two,

You will very much oblidge Sir,
Your most Humble and
Obedient Servant
MATTH STEWART.

P.S. Please direct for me at
Mr Will^{m} Sand's
Bookseller in Parliament Closs

To
MR MATTHEW STEWART
at Mr William Sand's Bookseller
in the Parliament Closs
Edinburgh.

GLASGOW 16*th Jan*^{r} 1741.

D^{R} MATTHEW

I am ashamed when I look upon the date of yours of Jan^{ry} 3^{d} that I have been so long in answering it, and am sorry I cannot this post send you a full account of what Mr Ja: Gregory has done in the problem of squaring the Hyperbola, for this is the 4th letter I have wrote this night, and have not time before the post goes of to send you it, but if you cannot find the book by the directions I now give you let me know by monday's post, and I shall write a letter that will procure you a sight of it. The treatise de Vera Quadratura Circuli et hyperbolæ is reprinted in the 2^{d} Vol. of Hugenii opera Varia. and Guido Grando's Theorematum Hugenianorum Demonstratio is in the 1st Vol. of Hugenii opera reliqua *i.e.* in the 3^{d} volume of the four which contain all his works. this last I mean Grando's book I believe contains a good deal more of what you want to see than Gregory's. Huygens works being but

lately printed I cannot but think you will meet with them in some Booksellers shop, if not I desire you may go to Mr Tho. Ruddiman keeper of the Advocates Library and give my service to him, and tell him I desire the favour of his letting you see both these books. if you get them not thus, I shall upon your letting me know it, send you in a letter that will obtain a sight of them. fail not to let me hear from you, I have no time to write any thing about your Propositions, but that the mutual relation of the Hyperbola and Logarithmick Curve you will find fully in Huygens theorems above mentioned.

I am

Dear Matthew

Yours affectionately

ROB: SIMSON.

To

MR MATTHEW STEWART

at Mr William Sands Bookseller

in the Parliament Closs

Edinburgh.

GLASGOW 25th Febr. 1741.

DEAR MATTHEW

I Received your letter of the 19th with the Answer to the Problem enclosed. I had seen the Problem about 2 weeks ago in the Magazine, and answered it upon the reading of it, without putting pen to paper, which was owing to my having long before this had the Loci, to which this is easily reduced. Your Lemma I demonstrated, but have not considered how it is to be applied, And indeed I could not so much as get time to write you this by monday's post, as I had fully designed.

You desire me to revise the Answer and return it with any alterations or Additions, but considering that there are only the bare Propositions : I see not what alterations you can mean except in the Stile; with respect to which I would not have you use the word reason but ratio. And the way of speaking major dato quam in ratione, is by no means intelligibly translated greater by a given space (or square) and in reason. for example your 5th Prop. which stands thus " If it were proposed that the Excess of the sum of the

"Squares of the lines drawn from the first above the sum of the "squares of the lines drawn from the 2^{d} may be greater or less, "than the sum of the squares of the lines drawn from the 3^{d} by a "given square and in reason" would be plainer in English thus viz. If it were proposed that the Excess of the sum of the squares of the lines drawn from the first above the sum of the squares of the lines from the 2^{d}; diminished or encreased by a given square may have a given ratio to the sum of the squares of the lines drawn from the 3^{d}.

Two of the following Propp. seem to differ only in this that what is called the first in one is the 2^{d} in the other and vice versa &c. I desire you will be so free with me as to let me know whether any person besides yourself is concerned in the Answer, or even in the Question; for if you be only concerned in them; I can give you some hints that may be worth your while. but if another person is really the proposer and that you are quite ignorant who he is, I would, if in your case, let him give in his own answer.

I have enclosed your paper, and shall be glad to hear from you with the first conveniency. I am in hast, wishing you all success in your inquiries

Dear Matthew

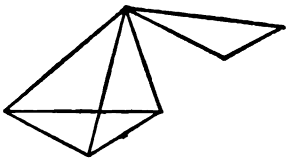
Your affectionate most humble

Servant

Rob: Simson.

Let me know what occasioned the bringing Polygons into the Problem since it would be more general, to have made use of points. and I suppose you know one of the Propositions in your answer is a Locus of Appollonius which has been shown several ways; That Locus of Apollonius I have long ago found out his own solution of, which I have shown to severals. be so good as to take no notice of the contents of this to any body. which I have observed with respect to your Answer most strictly.

If you will send your solution to the case of 3 points and 2 which is one of the simplest; I shall give my Judgment impartially with respect to the comparison of yours and my own.



I should be glad to see you soon here, but whenever you come be sure to call at me. adieu.

To
MR ROBERT SIMSON Professor
of Mathematics in the University
of Glasgow.

ED: *Feb.* 28. 1741.

SIR

I received yours of the 25th and am very much oblidgd to you for letting me know any alterations you thought proper to be made in the Answer I sent you last week.

There was one part of your Letter, Sir, which I do not fully understand namely, "Two of the following Propp: seem to differ only in this, that what is called the first in the one is the 2d in the other and *vice versa*, I wish you would let me know the particular Propositions, and likewise that one which you say is a locus of Appollonius. As I have never seen any of Appollonius's works I do not know which of them it is.

The method I take to find the Construction of the Problems containd in the Answer, is by reducing them to more simple Problems, which may be done by the Lemma in the Answer, the most of them for example may be reducd to this one.*
"Given two points in a plane from which there are drawn lines
" concurring in a point in the same plane that the square of one of
" the lines diminishd or encreasd by a given Square may be to the
" Square of the other line in a given ratio, requird the locus of the
" point where the lines concurr.

I would be glad to know your Construction of this Problem, and likewise your Demonstration of the Lemma, the Demonstration I have, being very tedious. I have sent, as you desird, my Solution to the case of 3 points and 2, but am very far from thinking that it will be in the least comparable to yours.

You desire me to be so free with you as to let you know if any other Person beside myself be concernd in the Answer or even in the Problem.

I sincerely declare, Sir, that there is no person but myself concernd either in the Answer or in the Problem, for I am not in the least acquainted with any person here that pursues the Study of Mathematics that I know off, nor did I ever exchange Letters with any person concerning that Study but yourself, Sir, And I should

* See Robert Simson's *Loci Plani* (Glasguae, 1749), p. 227.

be very glad how often I should have occasion to exchange Letters with you, Sir, as I am very sensible y^{t} you are both willing and capable to give me many usefull hints and Advices concerning that Study.

The occasion of bringing Polygons into the Problem and Answer, was, because I found some difficulty in the expressing of them when I made use of points. As I do not know when I shall go to Bute, I am very sorry that I have no prospect of seeing you soon, tho I am very desirous of having the pleasure of seeing you upon several accounts.

I am Sir Your most
Humble and Obedient
Servant MATTH STEWART

To
MR MATTHEW STEWART
at Mr William Sand's
Bookseller in Parliament Closs
Edinburgh

GLASGOW 13th *March* 1741.

D^{r} MATTHEW

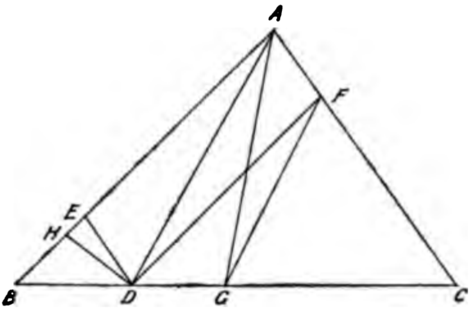
I designed to have wrote to you last post but was prevented. I have now come from a Company and sent you the demonstration upon the other side in a great hurry, So Excuse any thing amiss but send me a double of it when you have leasure for I have none *i.e.* no copy to my self. You see it agrees to both cases. the Demonstr. of the 1st sent formerly might also have been transferred to 2^{d} case, but I thought you would like this better. Your observation about the sum of the squares of lines drawn to the center is right, that sum being together with the multiple of the square of the semi diameter by the number of points equal to the given space. I cannot add a word more and have disobliged I fear those that I left who have sent for me half an hour ago, so must leave what else I was to say till next letter. Let me know if you easily and generally deduce the propertie of center of gravity by your Method. I am

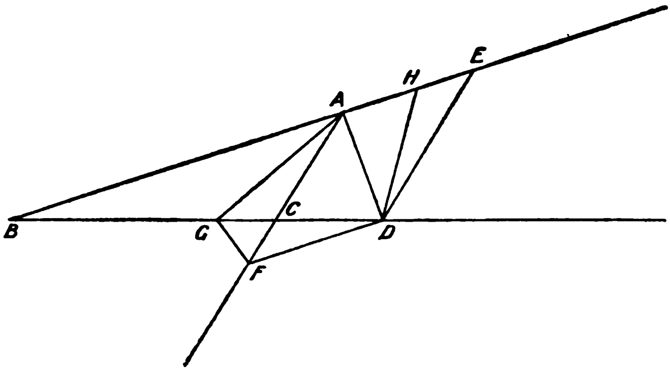
D^{r} Matthew

Yours affectionately
ROB: SIMSON

Let me hear from you
with first conveniency.

Prop.* Si a vertice A Trianguli ABC ducatur ad basim recta AD, a puncto vero D ducantur DE, DF lateribus AC, AB parallelae, ipsisque in E, F occurrentes; Erunt rectangula BAE, CAF simul aequalia quadrato ex AD et BDC rectangulo. Si vero ducatur AD ad basim productam, reliquis manentibus, erit excessus rectangulorum BAE, CAF aequalis excessui quadrati ex AD et rectanguli BDC.





Ad BC ducatur recta AG faciens angulum GAF aequalem ipsi ABC seu FDG; a puncto vero D ducatur ad BA recta DH faciens angulum ADH aequalem angulo AED, et in triangulis ADH, AED

* This is the first proposition of Matthew Stewart's *Some General Theorems* (Edinburgh, 1746). Stewart's demonstration is different from Simson's.

quae communem habent angulum DAH, erit reliquus angulus AHD aequalis reliquo ADE hoc est alterno angulo DAF. Quoniam vero aequales sunt anguli GAF, FDG erunt puncta G, D, A, F in circulo; quare junctâ GF, erit in Fig. 1. erit angulus exterior FGC aequalis interiori et opposito DAF in quadrilatero DAFG, hoc est angulo AHD; et propterea est angulus DGF aequalis ipsi DHB. In Fig. vero 2^{da} est angulus FGD, aequalis ipsi FAD, in eodem sc: segmento circuli, hoc est angulo AHD. Aequales igitur sunt anguli FGD, AHD in utraque figura, et in triangulis DGF, BHD aequales etiam, propter parallelas, sunt anguli FDG, HBD quare [4.6.] est BD ad BH, ut DF ad DG; et rectangulum BDG aequale erit rectangulo contento ipsis BH, DF, hoc est contento BH, AE. Et quoniam ex constructione angulus ABC aequalis est ipsi GAC, et, in triangulis BDE, ACG propter parallelas est angulus BDE aequalis angulo GCA; erit [4.6.] BD ad DE, ut AC ad CG; quare rectangulum contentum BD, GC aequale est contento AC, DE hoc est rectangulo CAF. Et ostensum fuit rectangulum rectangulum BDG aequale contento BH, EA; Ergo in figura I^{ma} rectangulum BDG una cum contento BD, GC, hoc est [1.2.] rectangulum BDC aequale est contento BH, EA una cum ipso CAF rectangulo. In triangulis autem AHD, ADE quoniam aequales sunt anguli ADH, DEA et communis DAH, erit rectangulum HAE aequale quadrato ex AD in utraque fig: Ergo, additis aequalibus, erit rectangulum contentum BH, EA, una cum ipsis CAF, HAE hoc est erunt rectangula BAE, CAF simul aequalia rectangulo BDC una cum quadrato ex AD in fig. sc: I^{a} In fig. vero 2^{da} est [1.2^{di}] rectang. BAE una cum ipso HAE aequale contento ipsis BH, AE hoc est (ut ostensum fuit) rectangulo BDG, hoc est ipsi BDC et contento BD, GC simul; hoc est ipsis BDC, CAF. Ergo BAE, HAE rectangula hoc est BAE rectangulum et quadratum ex AD aequale est ipsis BDC, CAF. et propterea excessus ipsorum BAE, CAF aequalis est excessui rectanguli BDC et quadrati ex AD.

Q.E.D.

To Mr Robert Simson Professor
of Mathematics in the University of
Glasgow.

Edr. *Mar* 28. 1741.

Sir

I received yours together with your Construction of the locus of Appollonius, which gave me a great deal of Satisfaction, as you was pleasd to let me know that you design soon to publish your treatise de locis planis, and that the Theorems I sent you might, if it was agreeable to me, appear to better advantage there than in the Magazine, You need not doubt, Sir, but that it will be very agreeable to me, as I am persuaded they can appear no where to so good advantage. I shall send you them and likewise the method I take to demonstrate them, as soon as I can, but you will find it absolutely necessary to put them in another form before they can deserve a place among any of yours. I will very cheerfully give you all the Assistance I can in Copying your treatise and drawing the Figures.

I find the Printers of the Magazine have got sent them an Answer to the Problem with some very severe reflections upon the Antients, particularly Appollonius, the Answerer observes that the analysis they had is but very lame and obscure in comparison of the modern Algebraic method, and affirms that they did not even fully understand any kind of analysis they had. I do not know if this Answer will be publishd, but if you please, Sir, I can privately procure you a Copy of it.

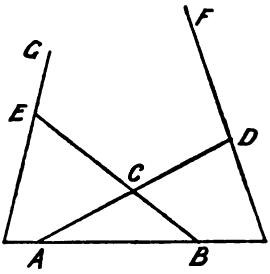
I am told that there is one Jack a Teacher of Mathematics in this Town about to publish a Translation of your Conics. I would be glad to know if he has your Concurrence.

I am Sir your most Humble
and Obedient Servt
Matth Stewart.

Sir.

I have sent you inclosd the analysis of the first of the loci mentiond in my last, upon which I found the second entirely

depended, the case when the two right lines given by position are not parallel, I found might be very easily deduc'd from the 44th and 45th of 5. Conics, but when the right lines given by position are parallel, does not appear to me to be so easily deduc'd. I have likewise sent you the analysis of this locus.



If from two given points A, B, there be drawn AC, BC to a point C and let AC, BC cut DF, EG two right lines given by position in D, E, such that the segment of the right line DF intercepted between D and the given point F may be to the segment of the line EG intercepted between E and the given point G in a given ratio, or that the rectangle of these segments may be given, requird the locus of the point C. By the method I took to analyse these loci I reducd them to the locus, of the Antients to four streight lines. After the same manner, when the rectangle ACB is to the rectangle DCE in a given ratio or the rectangle ACE to the rectangle BCD in a given ratio, I found the locus of the point C to be a Conic Section. I have sent you inclosd likewise the Construction I have of the Problem in the Magazine, [about a dozen of lines have been crossed out here] but will reserve the Construction of the other Loci I wrote you some time ago till I see you. I wish A. B were desird to send the Construction of the two Loci he mentions in the Conclusion of his paper, which he promises to do when desird, for it would appear I think, from the way he expresses them, that he mistakes them quite.

I would be very glad to have the pleasure of a Letter from you, with your Conveniency and to have your remarks upon the two inclosd papers.

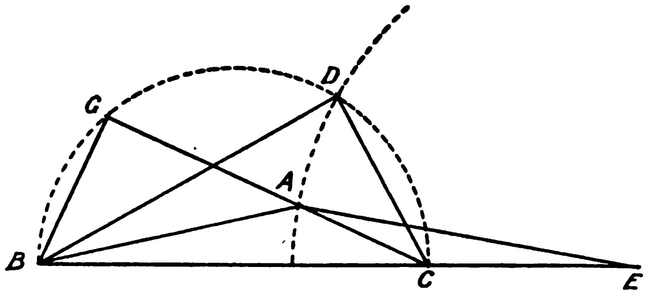
Mr Jack I am told, Sir, still continues his design, he gives out that he at first thought it would be agreeable to you, but that he now finds the Contrary, he proposes to make considerable Alterations.

I am Sir Your most humble
and obedient Servt MATTH
STEWART

EDr *July* 20 - 1741.

To

Mr Matthew Stewart
at Mr William Sands's bookseller
in the Parliament Closs
Edinburgh.



Sir

Glasgow 29th Novr 1742.

I got your letter of the 20th Current, for which I am obliged to you, and am well pleased with your account of the reason of not writing sooner, and that Mr M'Laurin has received you kindly. I got no leasure to consider the Porism which you sent till saturday last. It is elegant and wants not difficulty as you observed. Pray let me have your construction of it by next letter. I here send you mine.

Let ABC be a given triangle (in position) &c. as in your letter. Upon BC the base make the Triangle BDC having a right angle at D, and its sides BD, DC in the same proportion with BA, AC (which may be done by making the angle CAE equal to ABC and describing from center E thro A a Circle meeting one described on BC as a diameter in D and joining BD, DC) D shall be the point sought, and, the perpendicular BG being drawn upon CA, the ratio sought shall be that of the square of BG to the square of BD, viz: this is the ratio the sum of the squares of the perpendiculars has to the square of the line drawn from the point in the base to the point D. I am in much haste

Dr Matthew

Yours affectionately

Rob: Simson.

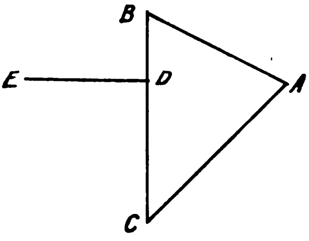
Ed^{r}. *Dec.* 2. 1742.

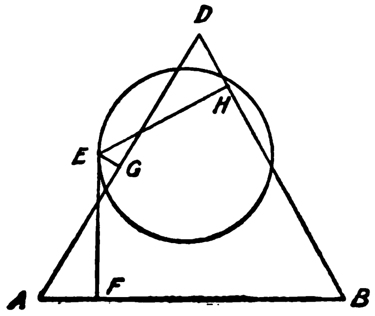
Sir

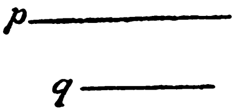
I had yours of Nov^{r} 29th for which I heartily thank you, I have sent, as you desird my construction of the Porism which differs very little from yours. I have likewise sent you another Porism, which to me appears to be more elegant than the first the construction of which I am not quite Master of at present but shall in my next, if desird, send you it. In my last I told you that I delayed writing to you till I would know if Mr M'Laurin was to have a fourth class. for he seem'd to be uncertain when I spoke with him, the reason why I thought he was then uncertain was that when I offered him money, he refusd it and told me, he would talk of that again with me after he had taken up his fourth class, accordingly I went to him, since I wrote you last, and offerd him money again, but was extremely surprised to find he would by no means take it, he told me I was very welcome to attend any of his classes I thought proper, but that he would take no money from me, I thought myself bound to write you this, as I am persuaded Mr M'Laurin's extraordinary civility to me must be entirely owing to you. I would be glad to hear from you soon. I am

Sir, your most Humble & obedient
Servant Matth Stewart.

In the base BC take the point D such that BD may be to DC as the square of AB to the square of AC at D erect DE perpendicular to BC and let the square of DE be equal to the rectangle BDC, E is the point given.







Porism. Let there be a circle and a right line AB both given by position, and let p, q be two given magnitudes. two right lines AD, BD are given by position, such that if from any point E in the circumference of the circle there be drawn perpendiculars to AB, AD, BD meeting AB, AD, BD in F, G, H the square of EF together with the space to which the sum of the squares of EG, EH has the same ratio that p has to q will be given.

but p must be greater than the half of q.

To
Mr Matthew Stewart
at Mr William Sands's bookseller
in the Parliament Closs
Edinburgh.

Glasgow 27th Decr 1742

Dr Matthew

P.S. Let me have any Mathematical news.

You may think I have too long delayed answering yours of the 2d of this month, but it is very seldom I can have the quiet that is proper for considering such questions as the last you favoured me with, and I liked not to write to you before I had considered it. It is a very elegant porism and wants not difficulty, when you send any such again you may send at the same time your solution of them which will save me time. I renew the desire I made to you when last here that you would not communicate these things till I see you at least; In the meantime I wish you would as your leasure can permit find out as many as may be of them, and be sure to write down both Analysis and composition, because when I publish an account of the Porisms I shall be glad to have your store to encrease mine which shall every one of them be particulary acknowledged in the book. Let me have your Construction by next post, and explain what you mean by saying you were not quite master of it when you wrote; be pleased to let me know if you have made it known to any. I doubt not the Construction I have given you upon the other side will please you. I have wrote the whole of this in company rather than delay it till another Post and am

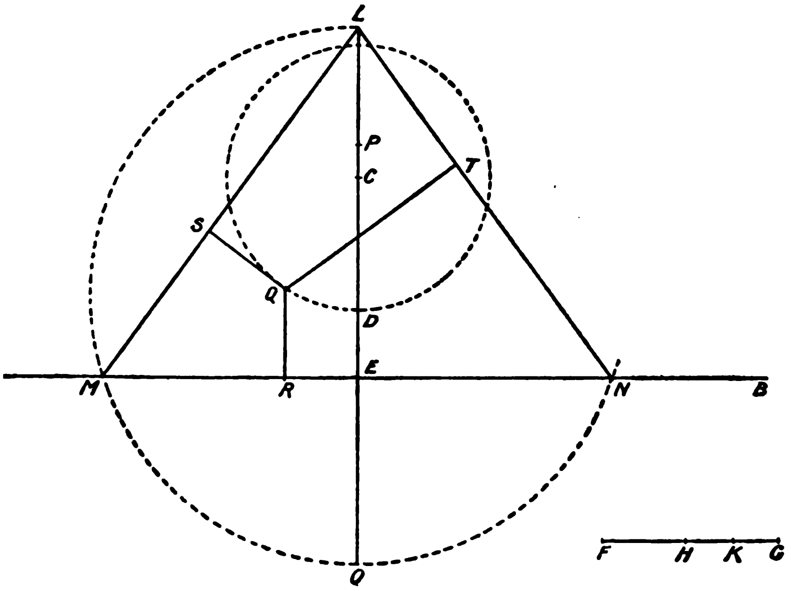
hamed to keep the gentleman that is with me any longer in iting. So can only add that the Porism may be made more nerall if proposed thus.

Having given in position a streight line and a circle and two gles A and B. There shall be given in position two streight lines ch that if from any point in the circumference there be drawn to ose lines, two others which make with each of the former the gle A, and a third line to the line given in position making with the angle B, the sum of the squares of all three shall be given.

I am

Yours affectionately ROB: SIMSON.

Data positione recta AB, et circulo, cujus centrum C et semidiameter positione dato, datâque ratione FG ad GH ; Dabuntur positione duae rectae LM, LN ad quas, ut et ad rectam AB ductis a quovis in circumferentia puncto Q perpendicularibus QR, QS, QT: Quadratum ex ea QR quae ad AB ducta est, una cum spatio quod ad summam quadratorum ex reliquis rationem habet eandem quam FG ad GH, aequale erit spatio dato.



Const. Ducatur ad AB perpendicularis CE (quae occurrat circulo in D) et bifariam sectâ HG in K, fiat ut FK ad HG, ita EC ad quartam CL quae ponatur in EC producta, in qua, ad contrarias partes, sumatur EO ipsi EC aequalis, et super diametrum LO descriptus circulus occurrat rectae AB in M, N punctis.
Junctae LM, LN erunt rectae quaesitae. Et in EC producta sumptâ CP tertia proportionali ipsis LC, CD; erit rectangulum LEP spatium quaesitum.

Excuse any errors may be upon the other side, for I have not time to examine it. also keep this leaf for I have no double of it, or of the Scheme.

Ed^r. *Dec.* 30. 1742.

Sir.

Yours of the 27^th instant came safe to hand which gave me a good deal of satisfaction, as I had long'd very much for an answer from you. When I mention'd in my last the Porism whose construction you was pleas'd to favour me with I had not the least intention of mentioning it to you by way of question. all I proposd to myself was to have your opinion of it, my not sending the construction of it at that time was purely owing to my not being then quite Master of it as you may easily perceive by my not mentiong a limitation which is absolutely necessary namely that the magnitude p must be greater than the half of q, this limitation I did not observe till after I had seal'd your Letter and had not then time to alter it, as my construction is almost the same with yours save only that yours exceeds mine somewhat in simplicity I think it needless to send you mine. You renew the desire you made to me when I saw you last that I would not communicate anything relating to Porisms to any here, this, Sir have religiously observ'd and will observe. You desire y^t I should investigate as many Porisms as I can and write down their Analysis and composition, and are pleased to signify to me your inclination to take notice of them in a way, which I own Sir I never expected, as I never had such an high opinion of myself as to think I was capable of investigating any thing that would deserve a place among yours, however, I shall endeavour to investigate as many as I can, and am hopefull to be in condition to investigate sixteen if not twenty that appear to me

intirely new, most of which I expect will exceed the one sent you last in difficulty and elegance, what makes me expect to be in condition to investigate so many is this, I have for some time past been thinking on several Propositions that to me appear to be intirely new, and was resolvd as soon as possible to put them in the best order I could and to publish some small thing that might be a specimen both to my Friends and others of my having dealt somewhat in this Study, I [did not incline to write you—*scored out*] was not resolv'd to Publish any of my Propositions in the form of Porisms, tho that was the form I thought several of them ought to appear in, but as you was pleasd to let me know that you had some thoughts of publishing something on this subject soon I was resolv'd as much as possible to avoid any thing that would seem to have the least tendency to this. I did not incline to write you [anything—*scored out*] concerning this my design till I had put my Propositions into order, and was resolv'd then if it would be acceptable to you that you would do me the honour to revise them, but upon receipt of your last I have laid this design intirely aside, but will endeavour as soon as possible to put them in the best order I can, that you may [pick—*scored out*] choose out such as you think proper for your design, the end I mentioned to you I propos'd to my self by publishing some small think, [*sic*] will be accomplishd by this in a manner beyond my expectation, only I must own Sir I would be extremely satisfay'd, were it agreeable and convenient to you that your Book were publishd as soon as possible not only with regard to myself, but to every one that has a regard for ancient Geometry a Book of this kind is very much wanted, and I am persuaded as you have long had the Publishing of this Book in your view, you can in a very little time prepare it for the Press. the design of the inclosd paper is this, Mr M'Laurin seem'd to insinuate to me that his civility to me which I mentioned to you in my last was owing to his having *had a character* of me, this occasiond my giving him this Paper, since I wrote you last, which he read over, and told me at the same time that he thought it would be worth my while to explain these things more fully. I did not give him in the least any hint of the method I had for explaining these things. I have not spoke with him since concerning this paper, I hope you will not think by my showing him this paper that I have acted contrary to the promise made to you, but

if you think proper I shall for the future observe a profound silence with regard to any things of this Nature. the first thing mentiond in the paper I told you of a twelvemonth ago.

I conclude this long and tedious Letter with earnestly entreating the favour of a Letter from by next post if possible which will be very acceptable to

Sir Your

P.S. Sir you'll please take care of the enclosd paper as I have no copy of my self and may perhaps incline to see it afterwards.

To

Mr Matthew Stewart
at Mr William Sands's bookseller
in the parliament closs
Edinburgh

Glasgow 3d. *Janr* 1743.

Dr Matthew

I received your agreeable letter in due time, but by reason of a faculty meeting on fridays night could not write a return. the hurry on new years day and another faculty this night, the papers for which I have but just now prepared, prevents me from writing now at the length I designed. I am glad you have so many Propositions which may be put into the form of Porisms, and wish to know in what manner you designed to have published those you speak of, because if you think it would be too long to defer till I could publish the account I am to give of the Porisms when yours might be published in the form of Porisms, (such I mean as properly could be brought into that form) they might be printed as an Appendix to the Loci plani in the form you had designed to print them in yourself, at the same time you might add the Locus you printed in the Magazine and such of those others connected with it as you thought fit. I want only the figures and to transcribe a few pages of the Loci plani to have them ready for the press, And design, if by any means I can prevail with you, to have you with me here as soon as you leave Edinburgh, till the book be printed. you need not be solicitous about expenses in staying here, which I shall make very easie to you. As to the keeping private what

relates to the porisms I am obliged to your prudence which I hope you will continue for reasons will satisfie you fully at meeting. In the meantime you will think it reasonable, how much soever I encline to communicate any small things I have that may be of any use to the publick, that I should be the first publisher myself; and this has made me keep the porisms to myself, Except the few hints Mr Ja: Moor, John Williamson and yourself got from me. Let me know whether of the 2 ways above you like best, or if this 3[d] will not be as good as either, viz: To print at the end of the Loci plani the paper you designed to print by itself, And to gather all the porisms you can find, & I shall publish these in the form of Porisms with those few of Euclid I have been able to discover, Mr Fermat's and my own, in the account I am to give of the Porisms which will be some small time after the Loci plani are published, I mean within a year or thereabouts. I am obliged to break of and have just time to add that I am exceedingly pleased with the propositions you enclosed some of which I saw in the reading how they might be done. they and the propositions on which they depend deserve well to be published both as they are elegant new and usefull. I am

D[r] Matthew

Your very affectionate
humble servant
ROB: SIMSON.

I had designed to mention the limitation you speak of in my last but forgot. the ratio viz: must be greater than that of 1 to 2. You need not be afraid of my losing the paper or any you have sent or shall send.

SIR.

I send you this in answer to yours of the 3[d] instant which came to me in due time, I am at a loss how to express my thankfulness to you for the kind offer you make me of printing the Papper I design'd to publish, at the end of your Loci plani by way of Appendix. I enclin'd Sir to publish some small thing as soon as possible, for a reason mentiond in a former Letter, and as you are pleasd to acquaint me that you want only the Figures and to transcribe a few pages of the Loci plani to have them ready for the press, and have made me the kind offer above mention'd I frankly

own Sir your kind offer is very agreeable to me. as to the manner I design'd to publish I was not fully determin'd till I would have finishd all I design'd to and was resolv'd to be directed by you then to publish in the manner you would think properest. I would be glad to know from you Sir, in what Volume you design to publish your Loci plani and if you design to have it printed here, if you be under no engagement to any printers here I would fain hope the two printers with whem my Brother in Law is in Company would serve you as well and as reasonably as any here, they were both with Mr Ruddiman when your Conics were a printing and had then in a manner the whole charge of the printing. Mr Jack inclind to have employd them to print his Book, but as I happen d to be in Town then and had got notice of his design, I told my Brother that I knew Mr Jack's design was intirely disagreeable to you, which occasiond their declining to be concernd in such an injurious design. You may perhaps Sir incline to have it printed at Glasgow I would be loth to insinuate any thing against the Glasgow printing as I very well know the Gentlemen you will in that event employ, they may indeed have as good types as the types here but it is plain that the Glasgow printers are not so much master of the press casting off as those here, and that a Book suffers as much by bad casting off at the press as by bad types, and I should be sorry that your Book should meet with any injustice this way as it is very probable that it will reach several places abroad [all the corners of Europe—*scored out*]. I hope Sir you will excuse me in the freedom I have used, I would not have wrote you this Sir were I not persuaded my Letter will fall into no hands but yours. You are pleas'd to write me, Sir, that you design'd if by any means you can prevail with me to have me at Glasgow with you till your Book be printed. you may assure yourself Sir there will be no occasion for Arguments to persuade me to undertake what is so agreeable to me as I can propose to myself no higher satisfaction than to have the pleasure of your Company for some time, and were I entirely at my own disposal you might command me when you will, but Sir I have reason to expect my friends incline I should be at home this Summer, for reasons I will acquaint you off at meeting, but as I encline very much to your proposal, and were it equally convenient for you that I should come to Glasgow in March or the beginning of April, I should be extremely satisfy'd, I enclin'd

indeed to attend Mr M'Laurins Lesson till the end but Sir I choose rather to loose his Lessons than be depriv'd of the pleasure I propose to myself by being at Glasgow with you. the reason why I propose to come to Glasgow so soon is that I expect my Father will agree sooner to this proposal of mine than to others, but I am even afraid he may even oppose this proposal of mine and can think of no better way to procure his consent than by your writting to him. I am persuaded a Letter from you will go as great a way to prevail with him as any thing can do. I would be glad to know if this be agreeable to you. I was designd to have wrote him this night, but shall delay till I have a return from you, which Sir I hope will be by this post if possible.

I am Sir Your most Humble and Obedient Servt

MATTH STEWART

EDR *Jan.* 6. 1743.

To

MR MATTHEW STEWART

at Mr William Sands's Book-

-seller in the Parliament Closs

Edinburgh.

GLASGOW 7th *Janr.* 1743.

DR MATTHEW

Your letter was most acceptable to me. I am as far from thinking ill of the friendly advice you give in relation to the printing of the Loci Plani, that I thank you kindly for having any of my concerns so much at heart. I design the book should be in Quarto tho' it should be never so thin, because I reckon that size most convenient especially in a book where there are Copperplates. As to the printing of it, were it to be at Edinburgh, the civility and kindness which the gentlemen who print for your Brother have shown me in the affair you mention would, as well as the regard you may be sure I have for any of your friends, induce me to employ them rather than others at Edinburgh. And tho' I have made no express promise to Mr Foulis here, yet having given him ground to expect the employment, I cannot handsomely without

just reasons give the printing to another, the rather that I know it is agreeable to my good friend Mr Hutcheson and that Mr Foulis and his Brother Mr Andrew have for a long time done every thing in their power to oblige me. As to the Casting off at the press, I know what you say is very true; that the workmen here have not been so much masters of it as at Edinburgh but I am told they have of late got some good pressmen here. I wish however you would desire the printers you mention to look into Mr Hutchesons book just now published, and get their opinion as to the print of it, for I think it looks pretty well and mine being to be in 4to I think might still look better, but if they think otherways let me know, because I design the book should be both exactly and neatly printed.

I forsaw the difficulty you tell me about your staying here any time; and in hurry when I wrote last forgot to offer you my service to write to your father about it, which I looked upon as the best method to procure the favour of your being allowed to stay here. I am glad you have the same thought. Pray let me know when it will be most proper, now, or some time after this to write to him, which I shall do in the most effectual manner I can. Let me also have his direction and how a letter may be safely sent to Bute, for there is nothing I long more for than to have you some time here.

There is one of the Loci plani of Apollonius viz: (the last of the 1st book as enumerated in Pappus) Si a quodam puncto ad positione datas duas parallelas ducantur rectae in datis angulis ita ut summa vel differentia specierum ex ipsis ductis aequalis fuerit dato spatio punctum illud continget rectam positione datam.* there is no difficulty when only two parallels are given, but when 3 or moe the determination of the Locus grows more operose as well as the Analysis, but the Determination chiefly. I have a particular solution for the case of 3 lines, and a general one for any number, but I thought them so long, that I resolved only to give the case of 2 parallels; if you find a little leasure any time I will be obliged to you if you can find a tolerably short and plain solution which I would insert in the book, for tho' in Dr Halleys translation of Pappus preface printed before his Edition of Apollonius de Sectione rationis there is ad positione datas duas parallelas, yet the number

* This is quoted from Halley's *Apollonius de Sectione Rationis* (Oxonii, 1706), p. xxxviii.

of parallels is not mentioned in the Greek, but give yourself no trouble about it except it be for your diversion till you come West.

I am
Dear Matthew

Yours affectionately
ROB: SIMSON.

No body shall know any thing of what you write about the printing. Nor do I show your letters to any whosoever. I forgot to tell you that if it could be as convenient for you May would be more convenient for me than April in regard I will be then quit of my classes at least about the middle of it.

SIR, You'll no doubt be somewhat surprised that I have not wrote you before now in answer to yours of the 7^{th} which came to me in due time. I was resolvd to have wrote you by last post but was disapointed for want of time, as I have been for the most part confined to my room since I was favoured with yours, having been somewhat indisposed. I have had no opportunity of desiring the printers to look into Mr Hutchesons Book and give their opinion as to the print of it, but as you seem to be pretty well pleased with the print of and are in a manner engag'd as to the printing of yours already I think it will be quite needless to propose this to them, the rather that I am apt to suspect they will be somewhat shy in an affair of this nature but if you think otherways I shall propose it to them: I am glad to find that you incline to write to my Father yourself to procure his consent to my being some time at Glasgow when I leave this, I am hopefull your Letter will prevail with him, you write me y^t if it would be as convenient for me, May would be more convenient for you, in regard you would be then quite of your classes, the reason why I proposd March or April was that either of these months was more convenient for me than may, but that I had reason to suspect my Father inclines I should be at home as soon as possible, when you write him, Sir, you may propose my coming in May, and if he oppose this which I am hopefull he will not, I will afterwards propose to him my coming in March or April if you incline, I wish you would write him concerning

this, as soon as you conveniently can, his name is Dugald you may direct for him Minister of the Gospel at Rothesay, Bute, Mr Rankin Tobacconist in the Briggate is a very proper hand to convey your Letter to him. I had almost forgot to write you that I had occasion to consider a Little the Proposition you mention in your last, when I was considdering the first Prop mentioned in the paper, I sent you some time ago, but will endeavour to considder it more fully before I see you, I was resolvd to considder the Proposition mention in that paper more fully in what I design'd should be publish'd, but you'll perhaps not think it so proper that any thing that relates to the Loci Solidi, should go along the Loci plani, and therefore if you you incline I shall drop this part of it, the rather that I am afraid it would swell my paper till a greater length than perhaps is convenient, I would be satisfyd to have your opinion concerning this,

ED^R. *Jan.* 13. 1743.

To

MR MATTHEW STEWART
at Mr William Sands's bookseller
in the Parliament Closs
Edinburgh

GLASGOW 28 *Jan^ry* 1743.

D^R MATTHEW

I got your letter of the 13^th current in due time and was sorry for the indisposition that confined you to your room and hope it is removed long before this. I believe I have reason both from my own experience, and from your great inclination to study, perhaps too uninterruptedly, to caution you to take care of your health by using proper recreation, such as walking abroad in the fields &c. which will enable you to return to your studies with fresh vigour. Continued studie especially in difficult things without diversion now and then cannot but be very prejudicial to health. when I received yours and some time after I was likewise indisposed and obliged to keep my room for some days but was not quite well during a forthnight and this is the reason that first kept me from answering yours

the day I received it, and to tell the truth when I grew better I forgot to write till now.

I agree with you that it is quite needless to ask the printers any question about Mr H —— n's book. I shall god willing write to your father very soon in the terms we have agreed on. I mean to let you stay here some time, and shall let you know his answer as soon as I get it. I am glad you had occasion to consider the Locus I wrote about, as to the Locus Solidus you mention I think you judge very right that it comes not in so naturally with the Loci plani, and therefor since you have enough beside it will be best to omit it. I have part of the Loci Solidi wrote, and if yours (the Locus Solidus) or any other you have or may find out happen not to be printed till I have mine ready we shall joyn them together. I have had a touch of the cholick which keeps me within today but is not so severe as to hinder me to write this. pray let me know by first post how you have been since you wrote, and give me any of your literary news. I am

Dear Matthew

Your affectionate humble

Servant

ROB: SIMSON.

SIR. I was favoured with yours of Jan 28, and would have wrote you a return by that post had I not been interrupted when about to write. I was sorry to find that you had been for some time indisposd and am hopeful that you are now perfectly recovered. I have been pretty well since I wrote you last I was resolvd to have wrote you a long letter now, and to have your opinion concerning the method I ought to follow in the analysing of Problems, but as I have not time just now to propose my difficultys, I shall write you again within a post or two. I have nothing further to write you now, but that I long much to hear how you are. I am

Sir, Yours &c

MATTH STEWART.

[This letter is not dated, but from Simson's reply it will be seen that it was either sent or received on 1st February, 1743.]

To

MR MATTHEW STEWART
to the care of Mr William Sand's
Bookseller in the Parliament closs
Edinburgh.

GLASGOW 24th *March* 1743

DEAR MATTHEW

Because in your last letter of the 1st of February, you wrote that in a post or two you would propose your difficulties about the method you were to follow in the Analysis of Problems, I did not send you an answer, expecting your letter very soon. but now I am afraid either that your health has not been as I wish or that some unforseen accident has hindered you. Let me know then by first post how all goes with you. And particularly take care and be cautious how you engage to agree to a proposal I believe has either already been or will soon be made to you about going to P——gh,* tho' the post is honourable enough, yet I would not wish any I have so great regard for as I have for you to be sudden in accepting of it. Mr Jo: Williamson, and I believe Mr James Muir too had that proposal made to them, but neither would accept; nor do I believe your father would agree to it. I have written to him and soon expect an answer which I shall let you know of as soon as I get it. pray let me hear from you

I am
Dear Matthew
Yours Affectionately
ROB: SIMSON

EDr. *Mar* 29. 1743.

SIR

I am very much asham'd, when I reflect that in my last to you, I promis'd to write you in a post or two, and have your direction how I was to proceed in the Analysing of Problems, and that I should have delay'd writing you so long. The true reason is this, when I wrote you last, I had then in view two Problems which I had been for some time before that thinking on, and when I attempted to solve them, I found them vastly difficult, and was very much at a stand what method to take to analyse them, which

* Petersburgh.

occasion'd my writting you in that manner, but upon second thoughts, I thought it better to delay considdering of them till I should be at Glasgow where I would have better access of letting you know my difficultys than by writting, especially as I foresaw they would cost me more time than I could well spare, for Mr M'L——n drives very furiously thro his Fluxions and I encline to keep as close pace to him as possible. I am very much oblidg'd to you for the insinuation you gave me in your concerning the proposal of going to P——gh. On this day three weeks ago, I was sent for by Mr M'L—— n, who upon my coming to him, show'd me a Letter he had got by that Day's post from London, concerning that affair, and when I had read it over, He made me an offer of that business in a very pressing manner. I was very much surpris'd at the offer, and propos'd to him some difficulties I had with respect to myself against accepting of that offer, He told me he would be against my accepting of it unless he got the terms settled in the best manner possible, and ample security that I should be at Liberty to come home when I encline'd after I had been three years there. I must own, as the affair was represented to me by him, I would have made no difficulty of accepting it, and told him so, that if my friends would agree to it, (which I was afraid they would not,) I would accept of it. He desir'd me to write as soon as possible to my Father concerning this affair, which I did in a very pressing manner, and soon had his return diswading me from it in the strongest terms imaginable. I told Mr M'L—— n so, and that as I found it was so disagreeable to My Father I did not incline to accept of it. He told me he was to make an offer of it to Mr Williamson or Mr Muir. He told me that he had wrote to his correspondent that he had me in view and that if my friends would give way to it, I would accept of it, and promisd to let me know when he got a return from his correspondent. This is all I know concerning this affair. I would be glad to know how you came to suspect the offer was either made or soon would be made me, and likewise your difficultys against accepting it. You will no doubt be very much surpris'd that in an affair of this kind I did not write you to have your opinion how I should behave, the truth is, I enclin'd very much to write you, but was diverted from it, for a reason I choose rather to acquaint you off when we meet, than write you. I would have wrote you this by last post in answer to yours but was interrupted when about to write you. I have kept

this affair as private as possible as I do not encline it should take air by my means. I would be glad to have a return from you as soon as you conveniently can.

This from Sir your most
Humble Serv^{t}.
M

To
MR MATTHEW STEWART
at Mr William Sands's Bookseller
in the Parliament Closs
Edinburgh.

GLASGOW 20th *April* 1743.

DEAR MATTHEW

I got your letter of March 29th in due time which I delayed to answer in expectation of getting one from your father which has within this hour come to my hand of the date 29th March in which very civilly he grants my request as you may see by the last sentence of it which I here have copied that you may fully know his mind, It is "As I would cheerfully goe into your desire as far "as is consistent with our affairs and views, so I shall be well pleased "how soon he comes home, That (if possible) matters may be so "concerted, as he may be in condition to return to you, and to his "power serve you for a few weeks, as your letter mentions." So you may take your own time, as far as is consistent with your fathers desire, of staying at Edinb^{r}: I shall be done with my classes in a few weeks and so in condition to use your assistance after you have seen your friends a few days at Bute.

I am glad you are not going to P—— gh. what relates to my coming to know of that affair and the reasons I have against it I shall defer till meeting. I am just now called upon to go out so wishing you may take sufficient care of your health while Mr M'Laurin drives on so fast and you consequently in a constant intensness of mind which needs more relaxation than I believe you use, I am

D^{r} Matthew
Your very affectionate
humble Servant
ROB: SIMSON.

31

To

Mr Matthew Stewart

To the care of the Reverend Mr Dougald

Stewart Minister at

Rothesay in

Bute.

Glasgow 5^{th} *August* 1743.

Dear Matthew

Mr White having with some strangers been at the College yesterday enquired earnestly at me for you, and how he could get a letter conveyed to Bute: I undertook to send it, so he has just now sent it to me. I believe it may come by some of Ascocks servants, John Rankin having told me he was to go for Bute tomorrow. this is all the occasion of my writing. I desire you may let me know how all affairs go with you. Tho Mr Moor has been a long time at the goat milk with Kilmarnocks family, and is still busied about them, he has got time to draw all the schemes of the 1^{st} 20 Propositions save 3, and sayes when he returns, which will be middle of next week he'l soon get the rest done; he went through the town to day with the E. of Kilnm: &c. and I believe will go to Buchanan with them. I have got all wrote but the 6^{th} of B. 2. the Cases of which rightly to distinguish, as I told you, was the only difficulty which I have now digested, but not wrote down. after you went off I vexed myself some further about the Proposition with the 8 cases, and have got it done in single lines in any easy way, tho subject to the varieties of comp. & divid. &c. I hope to see John Williamson here before he go for England, and would be glad it happened when you were here. I give my very kind respects to your Father and am ever

Dear Matthew

Yours affectionately

Rob: Simson

Pray let me know particularly how my Lord and Lady Bute & all the family are. adieu.

Sir. I long very much to hear how you are and to know how far the Loci plani are advanc'd in the Printing. I have been so much taken up since I saw you in preparing for our Presbytery w^e is to meet very soon, that I have had very little time to employ in Mathematical Enquirys. I have got a very simple Construction of this Problem, a Conic Section being given and two right lines AB, AC being given by position intersecting each other in the point A to draw a right line BC touching the Conic Section in the point D and meeting y^e right lines AB, AC in B, C such that BD may be to CD as the square of AB to the square of AC. I have not had time yet to write the Construction, however if you are desirous to see it I shall send you it in my next. Upon this problem depends the solution of this other problem a Conic Section being given, and any number of right lines likewise given by position to find a point in the Conic Section such that drawing from that point right lines in given angles to the lines given in position the sum of the squares of the lines drawn, may be a Minimum. I am persuaded such as deal in the Calculus would find insuperable difficulties in the solving of this problem even when there are but a few number of lines given by position. I have likewise hit upon a Locus Planus since I saw you, w^c I must own I like very well, it is this, Si in Semicirculo inscribatur quaevis Figura Aequilatera, et a puncto ducantur perpendiculares ad latera Figurae inscriptae, sitque summa quadratorum ex perpendicularibus aequalis spatio dato, Tanget punctum circumferentiam positione datam. I have not had time to write the Analysis and Composition of this Locus, yet, however, I know it will take up seven or eight pages of my write in Quarto, but if you are likewise desirous to see this Locus I shall endeavour to send you it. I have likewise met with a Porism, but as I have not yet fully considdered it I shall write you of this again

Roth. *Sept^r* 27. 1743.

33

To
Mr Matthew Stewart
at Rothesay in
Bute

Glasgow 30th *March* 1744.

My Dear friend

I got your letter of 23^{d} Feb^{r} only the day before I got Blairhall's. I caused James Robison give me notice of every boat going to Bute since I got it, but not getting yet a distinct answer from Wm. Miller about his going to Mr Steuart, I delayed writing till now when I was ashamed to delay longer. I have wrote Mr Stewart about his affair at length so need not repeat it. Daniel Monro came to me else perhaps I might have forgot to write in time by this boat. I am glad you are so near an end of your tryals to which I wish a prosperous issue. Mr Muir has sent in the schemes of the 1st book six weeks ago, and has dunned Smith who cuts them by several letters without getting a return. Mr Muir thinks he is ashamed to write him without sending the figures along: and fancies that he is employed by the master of Elphinstoun in graving some Mapps of the shires of the Lothian's and that this has kept him from cutting the figures. had you been in Glasgow all this while, I believe the book would have been printed a quarter of year ago, (but you need not speak of this). pray send me the Enunciation only of the last of the Porisms in the 3^{d} book, tho I reckon that the difficultest part of the Invention. I shall defer the news concerning Mr Leechmans affair till we see what the Synod does in it, Mr Leechman having last Presbytery day craved leave under a Protest and publick Instrument to complain to the Synod of the injurious treatment he has received from the presbytery, and besides the boat is ready to go off. I am charmed with your Property of the circle, but have not got a minutes leasure to think of it or any mathematical subject this long time. I long much to see you in the mean time refresh me by letters upon every occasion of a boat or by post. My kind respects to your father. in haste I am

Dear Matthew

Yours very affectionately

Rob: Simson

I had almost forgot to tell you that only on tuesday last Mr

Foulis's books arrived at Lieth, I mean that part of them in which the Mathematici Veteres are.

So there is yet time enough to get my Lord Butes mind about it.

To
MR MATTHEW STEUART
Professor of Mathematics in
the University of
Edinburgh

$\frac{28\ Nov^r}{\text{GLASGOW } 9\ Dec^r}$ N.S. 1752

DEAR SIR

The bearer of this John Coulter A.M. goes to Edinburgh to be Tutor to Sir Michael Steuarts sons, as he has read Mathematicks with me, and since by himself he is very desirous of the favour of your countenance and advice to both which I recommend him as I know he is a modest young man and wants much to improve himself in every usefull studie and particularly Mathematicks.

I shall be very glad to hear from you, and if you will not give me any literary news, let me know at least how you and your family and friends are. I am

Dear Sir

Yours affectionately
ROB: SIMSON

[The following is a draft of a letter which was in all probability intended for Maclaurin.]

SIR,

When I reflect on the Many and Distinguished tokens of Friendship you was pleas'd to discover towards me, when I had the pleasure of attending your Lessons, I am greatly difficulted how to Apologize for my Conduct in not writting you long before now, to testify in some measure the grateful sense I retain of these

undeserved Favours. Sometime after my return from your Lessons I was oblidg'd in order to gratify my Friends, to submit myself to undergo Trials in the Presbytery of Dunoon, and have only got them over a few weeks ago. As it is very likely from my present situation that I will be under a necessity of Preaching somewhat often, so I am afraid that I will find little or no time to employ in the Mathematical way.

I remember, Sir, while I attended your Lessons I had occasion to show you a small paper of mine, which with some other things contain'd this property of the Circle,

"Let there be any Regular Figure circumscrib'd about a Circle "and from any point in the Circumference of the Circle let there be "drawn perpendiculars to the sides of the Figure, twice the Sum of "the Squares of these perpendiculars will be equal to thrice the "multiple of the Square of the Semi-diameter of the Circle by the "number of the sides of the circumscrib'd Figure*."

As you was pleas'd to take greater notice of that Paper than I expected when I show'd it to you, and seem'd to be somewhat taken with this Property of the Circle, so this encourag'd me to considder these things more carefully afterwards, particularly the above mention'd Property of the Circle.

Sometime after I observ'd likewise this other Property of the Circle "That if there be any Regular Figure, but not a Triangle, "circumscrib'd about a Circle and from any point in the Circum-"ference of the Circle there be drawn perpendiculars to the sides of "the Figure, twice the sum of the Cubes of these Perpendiculars "will be equal to five times the multiple of the Cube of the Semi-"diameter of the Circle by the number of the sides circumscrib'd "Figure†."

When I had discovered this Property of the Circle and compard it with the other Property before mention'd, I suspected these two Propertys might be only particular Cases of some more general Property of the Circle, with respect to the higher Powers of the Perpendiculars drawn from any point in the Circumference of the Circle to the sides of the circumscrib'd Figure. After I had

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* See Matthew Stewart's *Some General Theorems*, pp. 20,21

† ,, ,, ,, ,, ,, ,, pp. 65,66

considderd this for some time, I at last discover d this genera Property of the Circle.

"Let there be any Regular Figure circumscrib'd about a Circle, "and let m be the Number of the sides of the Figure, and let n be "any integer number less than m; if from any point in the Circum- "ference of the Circle there be drawn Perpendiculars to the sides "of the circumscrib'd Figure, the sum of the n Powers of these "Perpendiculars will be equal to the n Power of the radius of the "Circle multiplied by *

$$m \cdot \frac{1.3.5.7.9.\ \&c.\ n \text{ terms}}{1.2.3.4.5.\ \&c.\ n \text{ terms}} \text{"}$$

I was resolv'd &c [See below]

From this General Property of the Circle I very soon observ'd that another Property as General and Elegant as this naturally flow'd namely, "Let there be any Regular Figure inscrib'd in a "Circle, and let m be the number of the sides of the inscrib'd "Figure, and let n be any integer number less than m, If from all "the angles of the inscrib'd Figure there be drawn right Lines to "any point in the Circumference of the Circle, the Sum of the "$2n$ Powers of these lines will be equal to the $2n$ Power of the "radius of the Circle multiplied by †

$$m \cdot 2^n \cdot \frac{1.3.5.7.9.\ \&c.\ n \text{ terms}}{1.2.3.4.5.\ \&c.\ n \text{ terms}} \text{"}$$

This Property of the Circle is but one case of another more general which is this, "Let there be any Regular Figure inscrib'd "in a Circle and let m be the number of the sides of the inscrib'd "Figure and let n be any integer Number less than m, and from "all the angles of the inscrib'd Figure let there be drawn right "lines to a point in the plane of the Circle and let the Sum of "the $2n$ Powers of these lines be invariable, the point in the plain "of the Circle will be in the Circumference of another Circle "concentric with the former. ‡

* See Matthew Stewart's *Some General Theorems* pp. 104, 105

† ,, ,, ,, ,, ,, ,, pp. 109–110

‡ ,, ,, ,, ,, ,, ,, p. 112

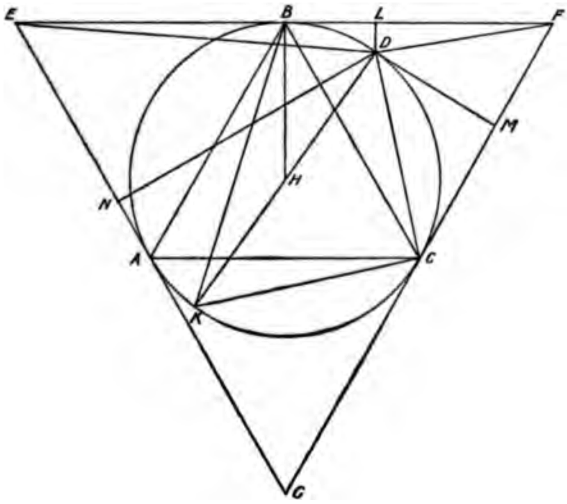
From this last Property naturally flows this other Property . . .

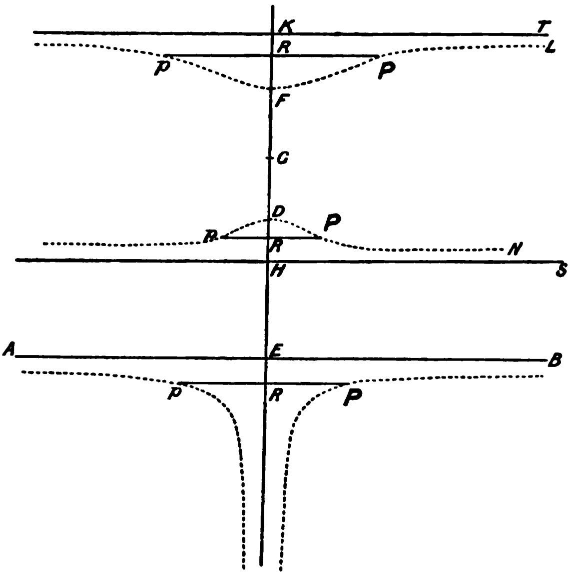
I was resolvd to have added some more Propertys of the Circle, but as this letter is of a greater length than I at first expected, I shall reserve these till I have occasion to write you again. I persuade myself from what I have already experienc'd of your goodness, that you will favour me with a return as soon as you conveniently can. And as this comes to you by my friend Mr Sands Bookseller in the Parliament closs you may commit your Letter for me to his care.

Please let me know how your Philosophical Society is like to go on, and how soon they propose to publish. We have been somewhat alarm'd here to hear that Mercury has been amissing for some time, but as this has not been taken notice of in any of the Publick Papers we saw here we do not know if we are to credit this report, I should be very fond to know the truth of this from you.

This is from D. S. with the greatest
Sincereity, Your most humble and
obed Servt

ROTH. 20th *May* 1744.





To

Mr Matthew Stewart Professor of Mathematicks in the University of
Edenburgh

Sir

The figure of which you gave me the construction, is drawn in a rude manner on the margin here. The lines are marked with the same letters as you did, so that you will understand the scheme without more words.

The figure has three Asymptotes, all parallel; and a fourth one KE cutting them at right angles. It consists of two conchoidal figures, and two hyperbolical ones. Any line such as pRP, parallel to the three Asymptotes, and cutting the curve in two points p & P, is bisected by the fourth Asymptote KE. So that the areas, in each figure, on different sides of the line KE are equal to one another.

Now I desire you will let me know what area you want, without asking one jot more or less than what is sufficient for your purpose. For when a curve is not quadrable, the whole area or different parts of it, are to be found by different artifices. The sum of two

areas is to be found by one rule, and their difference by another. And sometimes the sum or difference may be found, when the areas themselves cannot. Sometimes the whole area of a figure may be found with ease, when the parts of it cannot be assigned generally without great difficulty. For this reason I want to know your demand with all the limitations, it will admit of. Or if you will send me your probleme I shal be best able to judge of the limitations myself. As this is our post day, and I have several letters to write, I hope you will excuse this confused scrol, but I hope you will understand it.

Sir
Your most obedient
humble Servant

LEADHILLS 22 *May* 1755 JAMES STIRLING

Note on the different proofs of Fourier's Series.

By Dr H. S. Carslaw.

The Use of Green's Functions in the Mathematical Theory of the Conduction of Heat.

By Dr H. S. Carslaw.

§ 1.

The use of Green's Functions in the Theory of Potential is well known. The function is most conveniently defined, for the closed surface S, as the potential which vanishes over S and is infinite as $\frac{1}{r}$, when r is zero, at the point $P(x_0, y_0, z_0)$, inside the surface. If this is represented by G(P), the solution with no infinity inside S and an arbitrary value V over the surface, is given by

$$v = \frac{1}{4\pi}\int\int\frac{\partial}{\partial n}G(P) \,.\, V \,.\, dS,$$

$\frac{\partial}{\partial n}$, denoting differentiation along the outward drawn normal.

In the other Partial Differential Equations of Mathematical Physics similar functions may with advantage be employed, and,

in particular they have been found of great value in the discussion of the equation

$$(\nabla^2 + \kappa^2)u = 0.^*$$

It is the object of this paper to illustrate their use in the discussion of various questions in the Mathematical Theory of the Conduction of Heat. In this case *the Green's Function is taken as the temperature at* (x, y, z), *at the time* t, *due to an instantaneous point source generated at the point* $P(x_0, y_0, z_0)$, *at the time* τ, *the solid being initially at zero temperature, and the surface being kept at zero temperature.*

This solution may be written

$$u = F(x, y, z, x_0, y_0, z_0, t - \tau),$$

and u satisfies the equation

$$\frac{\partial u}{\partial t} = \kappa \nabla^2 u, \qquad (t > \tau).$$

However since τ enters only in the form $t - \tau$, we have also the equation

$$\frac{\partial u}{\partial \tau} + \kappa \nabla^2 u = 0. \qquad (\tau < t).$$

Further

$$\underset{t=\tau}{\text{Lt.}} \quad (u) = 0,$$

at all points inside S, except at the point (x_0, y_0, z_0), where the solution takes the form

$$\frac{1}{\left(2\sqrt{\pi\kappa(t-\tau)}\right)^3} e^{-\frac{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}{4\kappa(t-\tau)}}$$

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* *Cf.*

Pockels. *Über die Partielle Differential-gleichung* $(\nabla^2 + \kappa^2)u = 0.$ Theil IV. § 4. Leipzig 1891.

Schwarzschild. *Die Beugung und Polarization des Lichts durch einen Spalt.* Math. Ann. Bd. 55. 1902.

Finally, at the surface of S,

$$u = 0, \qquad (\tau < t).$$

Now let v be the temperature at the time t in this solid due to the surface temperature $\phi(x, y, z, t)$, and the initial temperature $f(x, y, z)$. Then v satisfies the equations

$$\frac{\partial v}{\partial t} = \kappa \nabla^2 v, \qquad (t > 0)$$

$$v = f(x, y, z), \qquad \text{initially, inside S,}$$

$$v = \phi(x, y, z, t), \qquad (t > 0), \text{ at surface of S};$$

and, since the instant τ of our former equations lies within the interval for t, these equations may also be taken as

$$\frac{\partial v}{\partial \tau} = \kappa \nabla^2 v, \qquad (\tau < t)$$

$$\text{and} \quad v = \phi(x, y, z, \tau) \quad \text{at the surface.}$$

Therefore we have

$$\frac{\partial}{\partial \tau}(uv) = u\frac{\partial v}{\partial \tau} + v\frac{\partial u}{\partial \tau},$$
$$= \kappa\left[u\nabla^2 v - v\nabla^2 u\right],$$

and

$$\int_0^{t-\epsilon}\left[\iiint \frac{\partial}{\partial \tau}(uv).\,dxdydz\right]d\tau = \kappa\int_0^{t-\epsilon}\left[\iiint (u\nabla^2 v - v\nabla^2 u)\,dxdydz\right]d\tau,$$

the triple integration being taken throughout the solid, and ϵ being any positive quantity as small as we please.

Interchanging the order of integration on the left-hand side, and applying Green's Theorem to the right-hand side, we have

$$\iiint [uv]_{\tau = t-\epsilon}\,dxdydz - \iiint [uv]_{\tau = 0}\,dxdydz$$
$$= \kappa\int_0^{t-\epsilon}\left[\iint\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)dS\right]d\tau$$
$$= \kappa\int_0^{t-\epsilon}\left[\iint v\frac{\partial u}{\partial n_i}\,dS\right]d\tau,$$

where $\frac{\partial}{\partial n_i}$ denotes differentiation along the inward drawn normal and we have used the fact that u vanishes at S.

On the right-hand side we may put $\epsilon = 0$, as there is no singularity in the integrand, and the left-hand side as ϵ approaches zero takes the value

$$[v_P]_t\left\{\iiint [u]_{\tau=t-\epsilon}dxdydz\right\} - \iiint [u]_{\tau=0}\cdot[v]_{\tau=0}dxdydz,$$

the first integral being taken through an element of volume including the point $P(x_0, y_0, z_0)$, where the infinity in u enters, and $[v_P]_t$ standing for the value of v at the point $P(x_0, y_0, z_0)$, at the instant t.

The choice of u, so that

$$\mathrm{Lt.}_{\tau=t}(u) = \mathrm{Lt.}_{\tau=t}\left(\frac{1}{[2\sqrt{\pi\kappa(t-\tau)}]^3}e^{-\frac{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}{4\kappa(t-\tau)}}\right),$$

makes the co-efficient of $[v_P]_t$ unity, *and we obtain*

$$[v_P]_t = \iiint [u]_{\tau=0}f(x, y, z)dxdydz + \kappa\int_0^t\left[\iint \phi(x, y, z, \tau)\frac{\partial u}{\partial n_i}dS\right]d\tau,$$

as the equation giving the temperature at $P(x_0, y_0, z_0)$, at time t, due to the initial distribution $f(x, y, z)$, and the surface temperature $\phi(x, y, z, t)$.

In the case of radiation at the surface, *the Green's Function u is taken as the temperature at (x, y, z), at time t due to an instantaneous source at (x_0, y_0, z_0) at time τ, the radiation taking place into a medium at zero temperature.*

The temperature at $P(x_0, y_0, z_0)$, at the time t due to an initial distribution $f(x, y, z)$, and radiation at the surface into a medium

at temperature $\phi(x, y, z, t)$, will then be found to be given by the equations,

$$[v_P]_t = \iiint u_{\tau=0} f(x, y, z)dxdydz + h\kappa \int_0^t \left[\iint u\phi(x, y, z, \tau)dS \right] d\tau,$$

$$= \iiint u_{\tau=0} f(x, y, z)dxdydz + \kappa \int_0^t \left[\iint \frac{\partial u}{\partial n_i}\phi(x, y, z, \tau)dS \right] d\tau,$$

the second of these equations being of the same form as that already obtained for the former case.

The use of Green's Function in the discussion of the equation of Conduction seems to have been noticed first by Minnigerode.* It is also developed in several papers by Betti, and is referred to in the other places noted below.†

The Green's Functions given in this paper in §§ 2, 5, may be written down by inspection, and the results obtained by the Synthetical Method * follow at once from our general theorems.

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* Minnigerode.
Über die Wärme-Leitung in Krystallen.
Diss. Göttingen. 1862.

† Betti.
(1) *Sopra la determinazione della temperatura variabile di un cylindro.*
Annali delle Università Toscane.
Tom. I. 1867.
(2) *Sopra la determinazione delle temperatura variabile di una lastra terminata.*
Annali di Matematica. Tom. I. 1867.
(3) *Sopra la determinazione delle temperatura nei corpi solidi ed omogenii.*
Mem. della Soc. Italiana delle Scienze.
Ser. III. Tom. I. 1868.
(4) *Sopra la propagazione del calore.*
Chelini Collezione 1881.

Sommerfeld.
Zur Analytische Theorie der Wärme-Leitung.
Math. Ann. Bd. 45. 1894.

Weber-Riemann.
Die Partiellen Differential-gleichungen der Physik.
Bd. II., § 51. 1901.

In the other cases, §§ 3, 4, 7, 8, these functions are obtained by the aid of Contour Integrals, following the method given by Dougall in his papers in these *Proceedings*.† This method is one of considerable power, and may be applied to many other problems of Mathematical Physics.

§ 2

LINEAR FLOW OF HEAT.

SEMI-INFINITE SOLID BOUNDED BY PLANE $x=0$.

In this case our general result is simplified by the consideration of the plane source over $x=x_0$, instead of the point source at (x_0, y_0, z_0), and the Green's Function is to vanish at the surface, and become infinite for $x=x_0$, at $t=\tau$, in the form

$$\frac{1}{2\sqrt{\pi\kappa(t-\tau)}}e^{-\frac{(x-x_0)^2}{4\kappa(t-\tau)}}.$$

When the solid is bounded by $x=0$, but is unlimited in the direction $x>0$, this function is clearly given by

$$u=\frac{1}{2\sqrt{\pi\kappa(t-\tau)}}\left[e^{-\frac{(x-x_0)^2}{4\kappa(t-\tau)}}-e^{-\frac{(x+x_0)^2}{4\kappa(t-\tau)}}\right],$$

and the solution of the problem, when the initial temperature is $f(x)$ and the boundary is kept at $\phi(t)$, is given by

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* Hobson.

Synthetical Solutions in the Conduction of Heat.

Proc. Lond. Math. Soc. Vol. XIX. 1888.

† Dougall.

(i) *The Determination of Green's Function by means of Cylindrical or Spherical Harmonics.*

Proc. Edin. Math. Soc. Vol. XVIII. 1900.

(ii) *Note on the Application of Complex Integration to the Equation of the Conduction of Heat.*

Proc. Edin. Math. Soc. Vol. XIX. 1901.

$$v_P\Big]_t = \int_0^\infty u_{\tau=0} f(x)dx + \kappa\int_0^t \phi(\tau)\left(\frac{\partial u}{\partial x}\right)_{x=0} d\tau$$

$$= \frac{1}{2\sqrt{\pi\kappa t}}\int_0^\infty f(x)\left[e^{-\frac{(x-x_0)^2}{4\kappa t}} - e^{-\frac{(x+x_0)^2}{4\kappa t}}\right]dx$$

$$+ \frac{x_0}{2\sqrt{\pi\kappa}}\int_0^t \phi(\tau)\frac{e^{-\frac{x_0^2}{4\kappa(t-\tau)}}}{\sqrt{(t-\tau)^3}}d\tau.$$

This result is obtained by the Synthetical Method by the distribution of sources and sinks along the axis of x and of continuous doublets of strength $2\kappa\phi(t)$ at $x=0$.

The corresponding Green's Function for the case of radiation has been obtained by Bryan,* and also by the author,† by the method of Contour Integrals to be used later in this paper.

§ 3.

Linear Flow of Heat.

Finite Solid bounded by the planes $x=0$ and $x=a$.

To obtain the Green's Function for the solid bounded by the planes $x=0$, $x=a$, we proceed from the solution

$$v = \frac{1}{2\sqrt{\pi\kappa t}}\left[e^{-\frac{(x-x_0)^2}{4\kappa t}} - e^{-\frac{(x+x_0)^2}{4\kappa t}}\right]$$

which satisfies the conditions at $x=x_0$ and $x=0$. This may be written

$$v = \frac{1}{2\pi}\left[\int_{-\infty}^{\infty} e^{-\kappa a^2 t}\cos a(x-x_0)da - \int_{-\infty}^{\infty} e^{-\kappa a^2 t}\cos a(x+x_0)\right]da.$$

* Bryan.
An Application of the Method of Images to the Conduction of Heat. Proc. Lond. Math. Soc. Vol. XXII. 1891.

† *A Problem in Conduction of Heat.* Phil. Mag. July 1902.

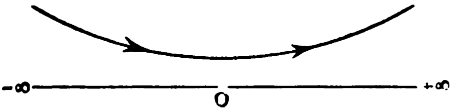
We replace these real integrals, by complex integrals in the a - plane, and obtain

$$v = \frac{1}{i\pi}\int e^{-\kappa a^2 t} \sin a x_0 \, e^{-iax} \, da, \qquad x > x_0$$

$$v = \frac{1}{i\pi}\int e^{-\kappa a^2 t} \sin a x \, e^{-iax_0} \, da, \qquad x < x_0,$$

the integrals being taken over the path (P) Fig. (1) in the a – plane, and the phase of a lying between 0 and $\frac{\pi}{4}$ to the right, and $\frac{3\pi}{4}$ and π to the left, at infinity.

FIGURE 1.



The path P in a-plane.

If we call this solution V, we must now choose a solution V_1, which will satisfy the conditions at $x = a$.

In this case we take

$$V_1 = -\frac{1}{i\pi}\int e^{-\kappa a^2 t} \frac{\sin a x_0 \sin a x}{\sin a a} e^{iaa} \, da,$$

over the same path (P), and we have now to examine the solution

$$v = V + V_1$$

$$= \frac{1}{i\pi}\int e^{-\kappa a^2 t} \frac{\sin a x_0 \sin a(a - x)}{\sin a a} da \qquad x > x_0$$

$$= \frac{1}{i\pi}\int e^{-\kappa a^2 t} \frac{\sin a x \sin(a - x_0)}{\sin a a} da \qquad x < x_0$$

which we shall show satisfies all the conditions of the problem.

Initial Conditions.

We have seen that $\underset{t \to 0}{\text{Lt.}}$ (V) has the form required by the Green's Function; we have thus to show that $\underset{t \to 0}{\text{Lt.}} (V_1) = 0$.

When we put $t=0$ in the integrand in V_1 the expression vanishes: for

$$\int \frac{\sin a x_0 \sin a x}{\sin a a} e^{i a a}\, da$$

has no singularity in the a-plane *above* (P) and the integrand vanishes at infinity when the imaginary part of a is positive provided

$$x + x_0 - 2a < 0.$$

Also the presence of the factor $e^{-\kappa a^2 t}$ causes the integral over the path (P) to converge uniformly towards its value for $t=0$, and we are thus entitled to take

$$\underset{t=0}{\text{Lt.}} \; [V_1] = 0.$$

Boundary Conditions.

These have been already satisfied by our choice of V_1 and it is clear that the two expressions

$$v = \frac{1}{i\pi}\int e^{-\kappa a^2 t} \frac{\sin a x_0 \sin a(a-x)}{\sin a a}\, da \qquad x > x_0$$

$$= \frac{1}{i\pi}\int e^{-\kappa a^2 t} \frac{\sin a x \sin a(a-x_0)}{\sin a a}\, da \qquad x < x_0$$

vanish at the boundaries $x=0$ and $x=a$.

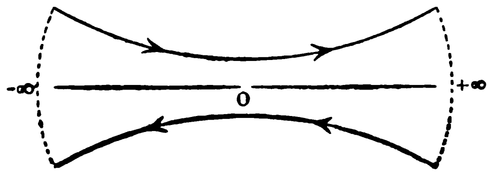
Hence the temperature at x, at time t, due to a source at x_0, at $t=0$, is given by these integrals over the path (P), and since the integrand is an odd function of a we may replace them by the forms

$$\frac{1}{2i\pi}\int e^{-\kappa a^2 t} \frac{\sin a x_0 \sin a(a-x)}{\sin a a}\, da \qquad x_0 < x < a$$

$$\frac{1}{2i\pi}\int e^{-\kappa a^2 t} \frac{\sin a x \sin a(a-x_0)}{\sin a a}\, da \qquad 0 < x < x_0$$

over the path (Q) of figure (2).

FIGURE 2.



The path Q in a-plane.

Expansion in Series.

From this result we deduce the expression for the temperature due to a source at $x = x_0$, in the form of an infinite series. Using Cauchy's Residue Theorem, since the singularities occur along the real axis at $a = \frac{n\pi}{a}$, our expressions become

$$\frac{2}{a}\sum_1^\infty \sin\frac{n\pi}{a}x \sin\frac{n\pi}{a}x_0\, e^{-\frac{\kappa n^2\pi^2}{a^2}t}.$$

Hence the Green's Function for this case is

$$u = \frac{2}{a}\sum_1^\infty \sin\frac{n\pi}{a}x \sin\frac{n\pi}{a}x_0\, e^{-\frac{\kappa n^2\pi^2}{a^2}(t-\tau)};$$

and the temperature at x_0 at time t when the initial temperature is $f(x)$ and the boundaries are kept at $\phi_1(t)$ and $\phi_2(t)$, is given by

$$v = \frac{2}{a}\sum_1^\infty \sin\frac{n\pi}{a}x_0 \int_0^a \sin\frac{n\pi}{a}x f(x) e^{-\kappa\frac{n^2\pi^2}{a^2}t}\, dx$$

$$+\frac{2\kappa n\pi}{a}\sum_1^\infty \sin\frac{n\pi}{a}x_0 \int_0^t [\phi_1(\tau) - (-1)^n\phi_2(\tau)]\, e^{-\kappa\frac{n^2\pi^2}{a^2}(t-\tau)}\, d\tau.$$

In the Synthetical Method this result is obtained by the distribution of sources and doublets along the axis.

§ 4.

Linear Flow.

Finite Solid: Radiation at Boundaries $x = 0$, and $x = a$ into a medium at zero.

The Green's Function in this case is obtained in a similar fashion. Starting with the solution

$$\mathrm{V} = \frac{1}{2\sqrt{\pi\kappa t}} e^{-\frac{(x-x_0^2)}{4\kappa t}}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\kappa a^2 t} \cos a(x - x_0)\, da.$$

we transform it into the integrals over the path (P) in the a - plane

$$\frac{1}{2\pi}\int e^{-\kappa a^2 t}\, e^{ia(x-x_0)}\, da, \qquad x > x_0,$$

or

$$\frac{1}{2\pi}\int e^{-\kappa a^2 t}\, e^{-ia(x-x_0)}\, da, \qquad x < x_0.$$

Associate with this solution, another given by

$$V_1 = \frac{1}{2\pi}\int e^{-\kappa a^2 t}\left[Ae^{iax} + Be^{-iax}\right] da$$

over the path (P) and determine A and B as functions of (a) by the boundary conditions

$$\mp\frac{\partial v}{\partial u} + hv = 0, \qquad \text{at } x = 0 \text{ and } x = a.$$

In this way we obtain

$$A = -(h + ia)\frac{h\sin a(a - x_0) + a\cos a(a - x_0)}{(h^2 - a^2)\sin aa + 2ah\cos aa}$$

$$B = -(h + ia)\frac{h\sin ax_0 + a\cos ax_0}{(h^2 - a^2)\sin aa + 2ah\cos aa}\, e^{iaa}$$

and

$$V + V_1 = -\frac{i}{\pi}\int e^{-\kappa a^2 t}\frac{(h\sin ax_0 + a\cos ax_0)(h\sin a(a - x) + a\cos a(a - x))}{(h^2 - a^2)\sin aa + 2ah\cos aa}\, da,$$

while, when $x < x_0$, we interchange x and x_0 in this expression.

Initial Conditions.

We have chosen V to satisfy the condition at $t = 0$ of the source at $x = x_0$: hence we have only to prove that

$$\underset{t=0}{\text{Lt.}}\ (V_1) = 0.$$

From the form of the expression for V_1 it will be seen that the singularities enter only at the roots of the equation

$$(h^2 - a^2)\sin aa + 2ah\cos aa = 0.$$

These are real and simple and there are thus no poles above the path (P). Also by examining the expression for V_1 it will be

seen that this vanishes at infinity in the upper part of the plane, provided that

$$x + x_0 > 0$$

$$x + x_0 - 2a < 0,$$

conditions which are both satisfied.

Hence when we put $t = 0$ in V_1, its value is zero, and the convergency factor $e^{-\kappa a^2 t}$, and the choice of the path P, cause the integral to converge uniformly to its value for $t = 0$.

Hence $$\underset{t=0}{\mathrm{Lt.}}\,(V_1) = 0.$$

Boundary Conditions.

The choice of A and B causes the conditions at $x = 0$ and $x = a$ to be satisfied.

Expansion in Series.

Taking the form for $x > x_0$,

$$v = -\frac{i}{\pi}\int e^{-\kappa a^2 t}\frac{(h\sin ax + a\cos ax_0)(h\sin a(a - x) + a\cos a(a - x))}{(h^2 - a^2)\sin aa + 2ah\cos aa}da,$$

over the path (P), we obtain

$$v = -\frac{i}{2\pi}\int e^{-\kappa a^2 t}\frac{(h\sin ax_0 + a\cos ax_0)(h\sin a(a - x) + a\cos a(a - x))}{(h^2 - a^2)\sin aa + 2ah\cos aa}da,$$

over the path (Q),

$$= 2\sum e^{-\kappa a^2 t}\frac{(h\sin ax_0 + a\cos ax_0)(h\sin ax + a\cos ax)}{a(h^2 + a^2) + 2h},$$

the summation being taken over the positive roots of the equation

$$(h^2 - a^2)\sin aa + 2ah\cos aa = 0.$$

The symmetry of this result shows that the expression also holds for $0 < x < x_0$, since in that case we had only to interchange x and x_0 in our former work.

Hence the Green's Function for this solid is given by the equation

$$u = 2\,\Sigma\,\frac{(h\sin\alpha x_0 + \alpha\cos\alpha x_0)(h\sin\alpha x + \alpha\cos\alpha x)}{\alpha(h^2+\alpha^2)+2h}\,e^{-\kappa\alpha^2(t-\tau)}.$$

The solution for an arbitrary initial distribution $v = f(x)$ follows at once, and we obtain for the case of the medium at zero temperature,

$$v = 2\int_0^a f(x')\left[\Sigma\, e^{-\kappa\alpha^2 t}\,\frac{(h\sin\alpha x' + \alpha\cos\alpha x')(h\sin\alpha x + \alpha\cos\alpha x)}{\alpha(h^2+\alpha^2)+2h}\right]dx'$$

as the temperature at x at the time t. This admits of integration term by term and may be written

$$v = 2\,\Sigma\, e^{-\kappa\alpha^2 t}\,\frac{(h\sin\alpha x + \alpha\cos\alpha x)}{\alpha(h^2+\alpha^2)+2h}\int_0^a f(x')[h\sin\alpha x' + \alpha\cos\alpha x']\,dx'.$$

It follows that

$$f(x) = 2\ \underset{t=0}{\text{Lt.}}\ \Sigma\, e^{-\kappa\alpha^2 t}\,\frac{(h\sin\alpha x + \alpha\cos\alpha x)}{\alpha(h^2+\alpha^2)+2h}\int_0^a f(x')[h\sin\alpha x' + \alpha\cos\alpha x']dx',$$

when $0<x<a$.

This expansion differs from that obtained by the Fourier Method * by the presence of the Convergency Factor $e^{-\kappa\alpha^2 t}$, and in the above proof we are not at liberty to proceed to the value $t=0$, the expansion occurring only as the limit when $t=0$. For the discussion of the convergency of the series *when* $t=0$, reference may be made to the two dissertations noted below.†

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* Fourier's Heat. Chapter V., Section I.
Kirchhoff. *Vorlesungen über Mathematische Physik*, Bd. IV., pp. 30-33.

† Knake. *Über die Wärme-bewegung in einem von zwei parallelen Wänden begrenzten Korper dessen Begrenzungen mit einem Gase in Berührung stehen.*
Diss. Halle. 1871.

Fudzisawa. *Über eine in der Wärme-Leitungs-Theorie auftretende, nach den Wurzeln einer transcendenten Gleichung fortschreitende, unendliche Reihe.*
Diss. Strassburg. 1886.

§ 5.

Two Dimensional Problems.

In the cases where the equation of conduction reduces to

$$\frac{\partial v}{\partial t} = \kappa\left[\frac{\partial^2 v}{\partial^2 x} + \frac{\partial^2 v}{\partial y^2}\right].$$

we use as Green's Function u the temperature at (x, y) at time t due to a Line Source generated at the instant τ along $x = x_0$, $y = y_0$, the surface being in the one case kept at zero, and in the other radiation taking place into a medium at zero.

With these values for the Green's Function, the temperature at $P(x_0, y_0)$ at time t, when the initial temperature is $f(x, y)$ and the boundary is either kept at $\phi(x, y, t)$ or radiation takes place into a medium at that temperature, is given by the equation

$$[v_P]_t = \int\!\!\int u_{T=0} f(x, y)dxdy + \kappa\int_0^t \left[\int \frac{\partial u}{\partial n_i}\phi(x, y, \tau)ds\right]d\tau,$$

integration taking place along the bounding arcs.

By means of this result we are able to write down the solutions of the two problems in which the solid is bounded by the plane $y = 0$, and extends to infinity in the direction $y > 0$: the initial temperature is $f(x, y)$: and, in the first case, the boundary $y = 0$ is kept at temperature $f(x, t)$, while in the second, radiation takes place into a medium at that temperature.

When the boundary is kept at temperature $F(x, t)$, the Green's Function is obviously given by

$$u = \frac{1}{4\pi\kappa(t-\tau)}\left[e^{-\frac{(x-x_0)^2+(y-y_0)^2}{4\kappa(t-\tau)}} - e^{-\frac{(x-x_0)^2+(y+y_0)^2}{4\kappa(t-\tau)}}\right],$$

and

$$\left[\frac{\partial u}{\partial n_i}\right] = \left[\frac{\partial u}{\partial y}\right]_{y=0} = \frac{y_0}{4\pi\kappa^2(t-\tau)^2}e^{-\frac{(x-x_0)^2+y_0^2}{4\kappa(t-\tau)}}.$$

Hence $[v_P]_t$

$$= \frac{1}{4\pi\kappa t}\int_{-\infty}^{\infty}\int_0^{\infty} f(x, y)[e^{-\frac{(x-x_0)^2+(y-y_0)^2}{4\kappa(t-\tau)}} - e^{-\frac{(x-x_0)^2+(y+y_0)^2}{4\kappa(t-\tau)}}]dxdy$$

$$+\frac{y_0}{4\pi\kappa}\int_0^t\int_{-\infty}^{\infty}\frac{F(x, \tau)}{(t-\tau)^2}e^{-\frac{(x-x_0)^2+y_0^2}{4\kappa(t-\tau)}}dx\,d\tau$$

gives the temperature at $P(x_0, y_0)$ at the time t.

In the case of radiation the Green's Function is given by Bryan,*

$$u = \frac{1}{4\pi\kappa(t-\tau)}\left[e^{-\frac{(x-x_0)^2+(y-y_0)^2}{4\kappa(t-\tau)}} - e^{-\frac{(x-x_0)^2+(y+y_0)^2}{4\kappa(t-\tau)}}\right.$$
$$\left. - 2h\int_0^{\infty} e^{-h\eta}\, e^{-\frac{(x-x_0)^2+(y+y_0+\eta)^2}{4\kappa(t-\tau)}}d\eta\right],$$

and may be obtained also by the method followed by the author in the similar case of Linear Flow.

Hence

$$\left[\frac{\partial u}{\partial n_i}\right] = \left[\frac{\partial u}{\partial y}\right]_{y=0} = \frac{h}{4\pi\kappa^2(t-\tau)^2}\int_0^{\infty} e^{-h\eta}\, e^{-\frac{(x-x_0)^2+(y_0+\eta)^2}{4\kappa(t-\tau)}}(y_0+\eta)d\eta$$

and the solution of the general problem, when the initial temperature is zero, is given by the equation

$$[v_P]_t = \frac{h}{4\pi\kappa}\int_0^t\int_{-\infty}^{\infty}\int_0^{\infty}\frac{F(x, \tau)}{(t-\tau)^2}.e^{-h\eta}.e^{-\frac{(x-x_0)^2+(y_0+\eta)^2}{4\kappa(t-\tau)}}.d\tau dx d\eta$$

This agrees with the solution obtained by Hobson by the Synthetical Method.

§ 6.

The Circular Cylinder.

Before discussing the corresponding problems for the cylinder, it will be necessary to define the solutions of Bessel's Equation which we employ.

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* loc. cit. p. 427.

The *Bessel's Function of the First Kind* is, as usual, defined by the equation

$$J_n(z) = \sum_0^\infty (-1)^s \frac{z^{n+2s}}{2^{n+2s}\,\Pi(s)\,\Pi(n+2s)},$$

where to make the function uniform we have to restrict the complex variable to a complete revolution about the origin, and we assume that the argument of z varies from $-\frac{\pi}{2}$ to $\frac{3\pi}{2}$.

For the *Bessel's Function of the Second Kind*, Hankel * uses

$$Y_n(z) = \frac{2\pi e^{ni\pi}}{\sin 2n\pi}(\cos n\pi\, J_n(z) - J_{-n}(z)),$$

and he obtains the following expressions for the limiting values of these two solutions when z becomes infinite, the real part of z being positive:—

$$J_n(z) = \sqrt{\frac{2}{\pi z}}\cdot\cos\left\{z - (n+\tfrac{1}{2})\frac{\pi}{2}\right\}$$

$$Y_n(z) = \sqrt{\frac{2\pi}{z}}\,\frac{e^{ni\pi}}{\cos n\pi}\sin\left\{z - (n+\tfrac{1}{2})\frac{\pi}{2}\right\} \dagger$$

In this paper it is necessary to use as Second Solution a function which vanishes at the positive imaginary infinity. Hankel shows that the function

$$U_n(z) = \frac{\pi}{2\sin n\pi}(J_{-n}(z) - e^{-in\pi}J_n(z))$$

has the limiting value

$$\sqrt{\frac{\pi}{2z}}\, e^{-\frac{in\pi}{2}}\, e^{i\left(z+\frac{\pi}{4}\right)}$$

whether the real part of z be positive or negative, and it is obvious that this solution vanishes at the positive imaginary infinity.

We shall use this as our Bessel's Function of the Second Kind.

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* Hankel. *Die Cylinder-Functionen erster und zweiter Art.* Math. Ann. Bd. VI., p. 494 (3) and (4).

† loc. cit. pp. 496-7.

It will be seen that

$$U_n(z) = \frac{i\pi}{2} J_n(z) - \frac{\cos n\pi}{2e^{ni\pi}} Y_n(z)$$

and it is to be noticed that what we here write as Y_n is not the solution given under that symbol in Gray and Mathew's *Treatise on Bessel Functions.* These writers follow Neumann's Notation, and denote Hankel's Y_n by $\bar{Y}_n$. The relation which connects the two is given (p. 66) by :—

$$\bar{Y}_n(z) = Y_n(z) - (\log 2 - \gamma) J_n(z),$$

γ, being Euler's Constant.

It also follows from the definition of U_n that when the real part of z is positive

$$i\pi J_n(z) = U_n(z) - e^{in\pi} U_n(ze^{i\pi}).*$$

§ 7.

Infinite Circular Cylinder: $r = a$; boundary at zero temperature.

To obtain the Green's Function for this case we proceed from the solution

$$v = \frac{1}{4\pi\kappa t} e^{-\frac{r^2 + r'^2 - 2rr'\cos(\theta - \theta')}{4\kappa t}},$$

corresponding to a Line Source at $(r'\theta')$ in the infinite solid.

We transform this into

$$\frac{1}{2\pi}\int_0^\infty e^{-\kappa\lambda^2 t} J_0(\lambda R)\lambda d\lambda, \dagger$$

where $R^2 = r^2 + r'^2 - 2rr'\cos(\theta - \theta')$.

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* Reference might also be made to the discussion in Graf and Gubler's *Einleitung in die Theorie der Bessel'schen Functionen.* Erster Heft. Cf. pp. 34, 35, 82-86. Bern, 1898.

† Cf. Gray and Mathew's Treatise, p. 77 (158).

Since, by Neumann's Formula,

$$J_0(\lambda R) = J_0(\lambda r)J_0(\lambda r') + 2\sum_1^{\infty} J_m(\lambda r)J_m(\lambda r')\cos m(\theta - \theta')$$

our expansion becomes

$$\frac{1}{2\pi}\int_0^{\infty} \lambda e^{-\kappa\lambda^2 t}\left[J_0(\lambda r)\,J_0(\lambda r') + 2\sum_1^{\infty} J_m(\lambda r)\,J_m(\lambda r')\cos m(\theta - \theta')\right]d\lambda.$$

If we assume that this series is uniformly convergent and can be integrated term by term,* this expression may be written

$$\frac{1}{\pi}\sum_{m=0}^{\infty} a_m \cos m(\theta - \theta')\int_0^{\infty} \lambda e^{-\kappa\lambda^2 t}\,J_m(\lambda r)\,J_m(\lambda r')d\lambda,$$

where $a_0 = \dfrac{1}{2}$ and $a_m = 1$, $m \geqq 1$.

Now

$$\frac{1}{i\pi}\int_{-\infty}^{\infty} \lambda e^{-\kappa\lambda^2 t} J_m(\lambda r')\,U_m(\lambda r)d\lambda$$

$$= \frac{1}{i\pi}\int_0^{\infty} \lambda e^{-\kappa\lambda^2 t}\left(U_m(\lambda r) - e^{im\pi}\,U_m(-\lambda r)\right)J_m(\lambda r')d\lambda$$

$$= \int_0 \lambda e^{-\kappa\lambda^2 t}\,J_m(\lambda r')\,J_m(\lambda r)d\lambda$$

since

$$i\pi\,J_m(\lambda r) = U_m(\lambda r) - e^{im\pi}\,U_m(-\lambda r)$$

in this case.

Therefore

$$\int_0^{\infty} \lambda e^{-\kappa\lambda^2 t}\,J_m(\lambda r)\,J_m(\lambda r')d\lambda$$

$$= \frac{1}{i\pi}\int_{-\infty}^{\infty} \lambda e^{-\kappa\lambda^2 t}\,J_m(\lambda r')\,U_m(\lambda r)d\lambda$$

$$= \frac{1}{i\pi}\int \lambda e^{-\kappa\lambda^2 t}\,J_m(\lambda r')\,U_m(\lambda r)d\lambda, \qquad r > r'$$

the path of integration being now the path (P) of Fig. (1) in the plane of the complex variable λ: and we must interchange r and r' in this result when $r < r'$.

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* Cf. Sommerfeld. *Die Willkurlichen Functionen in der Mathematischen Physik*, §§ 7, 12.
Diss. Königsberg, 1891.

This follows by Cauchy's Theorem since there are no poles of the integrand inside the contour formed by the real axis, the dotted lines, and the path (P) fig. (1). Further the integrand vanishes over the dotted lines when the part of the path is taken at an infinite distance: * and the argument of λ on the path P at infinity, must, on the right, lie between 0 and $\frac{\pi}{4}$, and on the left between $\frac{3\pi}{4}$ and π, since otherwise the factor $e^{-\kappa\lambda^2 t}$ would become infinite.

We have thus transformed the expression for the source into an infinite series each of whose terms is an integral over the path (P) in the λ plane.

We denote this solution, as before by V, and have the equation.

$$V = \frac{1}{i\pi^2}\Sigma a_m \cos m(\theta - \theta')\int \lambda e^{-\kappa\lambda^2 t} J_m(\lambda r') U_m(\lambda r) d\lambda \qquad (r > r')$$

the integrals being over the path (P) and r, r' being interchanged, when $r < r'$.

To obtain the conditions at the boundary $r = a$, we associate with this solution, another, denoted by V_1, where

$$V_1 = \frac{1}{i\pi^2}\Sigma a_m \cos m(\theta - \theta')\int A\lambda e^{-\kappa\lambda^2 t} J_m(\lambda r') U_m(\lambda r) d\lambda \qquad (r > r'),$$

and choose the term A so that the Boundary Conditions are satisfied.

We find, at once,

$$A = -\frac{U_m(\lambda a)}{J_m(\lambda a)}$$

and putting

$$v = V + V_1$$

we obtain the solution of our problem in the form

$$v = \frac{1}{i\pi^2}\Sigma a_m \cos m(\theta - \theta')$$

$$\int \lambda e^{-\kappa\lambda^2 t}\frac{J_m(\lambda r')}{J_m(\lambda a)}(U_m(\lambda r) J_m(\lambda a) - U_m(\lambda a) J_m(\lambda r)) d\lambda$$

when $r > r'$,

the integrals being taken over the path (P).

We shall now show that this expression satisfies all the conditions of the problem and then obtain an infinite series to which it is equivalent.

* Cf. The approximate value given below for the Bessel's Functions.

Boundary and Initial Conditions.

The Boundary Conditions are satisfied by our choice of the supplementary function V_1: and we have only to show that

$$\underset{t=0}{\mathrm{Lt.}}\ (V_1) = 0,$$

since $\underset{t=0}{\mathrm{Lt.}}\ (V)$ satisfies the conditions for a source at (r', θ').

Hence we have to show that

$$\underset{t=0}{\mathrm{Lt.}} \int \lambda e^{-\kappa\lambda^2 t} \frac{J_m(\lambda r')\, U_m(\lambda a)\, J_m(\lambda r}{J_m(\lambda a)}\, d\lambda,$$

over the path (P), vanishes.

The limiting forms, when λ is very large and lies in the upper part of the plane, of the Bessel's Functions occurring in this expression are given as follows :—

$$\left.\begin{aligned}
J_m(\lambda r) &= \frac{1}{2\sqrt{\pi\lambda r}}\, e^{-i\xi r + i\left(m+\frac{1}{2}\right)\frac{\pi}{2}} . e^{r\eta} \\
J_m(\lambda r') &= \frac{1}{2\sqrt{\pi\lambda r'}}\, e^{-i\xi r' + i\left(m+\frac{1}{2}\right)\frac{\pi}{2}} . e^{r'\eta} \\
J_m(\lambda a) &= \frac{1}{2\sqrt{\pi\lambda a}}\, e^{-i\xi a + i\left(m+\frac{1}{2}\right)\frac{\pi}{2}} . e^{a\eta} \\
U_m(\lambda a) &= \sqrt{\frac{\pi}{2\lambda a}} . e^{i\xi a - i\left(m-\frac{1}{2}\right)\frac{\pi}{2}} . e^{-a\eta}
\end{aligned}\right\}\ \lambda = \xi + i\eta.$$

Thus

$$\frac{\lambda\, J_m(\lambda r')\, J_m(\lambda r)\, U_m(\lambda a)}{J_m(\lambda a)}$$

vanishes at the positive imaginary infinity, when $r + r' - 2a < 0$.

Also since the zeroes of

$$J_m(\lambda a) = 0$$

are real and simple, there are no poles of the integrand, above (P), and

$$\int \frac{\lambda J_m(\lambda r')\, J_m(\lambda r)\, U_m(\lambda r)}{J_m(\lambda a)}\, d\lambda$$

vanishes.

The presence of the factor $e^{-\kappa\lambda^2 t}$ and the choice of the path (P) cause the integral

$$\int e^{-\kappa\lambda^2 t} \frac{\lambda J_m(\lambda r')\, J_m(\lambda r)\, U_m(\lambda a)}{J_m(\lambda a)} d\lambda$$

to converge uniformly to its value for $t = 0$, and thus

$$\underset{t=0}{\mathrm{Lt.}} \int \lambda e^{-\kappa\lambda^2 t} \frac{J_m(\lambda r')\, J_m(\lambda r)\, U_m(\lambda a)}{J_m(\lambda a)} d\lambda$$

vanishes.

The Initial and Boundary Conditions are thus both satisfied by the expressions we have obtained.

Expansion in Series.

We may replace the term

$$\frac{1}{i\pi^2} \int \lambda e^{-\kappa\lambda^2 t} \frac{J^m(\lambda r')}{J_m(\lambda a)} [U_m(\lambda r)\, J_m(\lambda a) - U_m(\lambda a)\, J_m(\lambda r)]\, d\lambda$$

over the path (P) by half this integrand over the path (Q), the integrand being a uniform, odd function of λ.

The poles of the integrand are the zeroes of $J_m(\lambda a)$, which lie symmetrically along the real axis and are not repeated.

Thus from this term in v we obtain $-\frac{1}{\pi}$ [sum of the residues along the real axis] which reduces to

$$\frac{2}{\pi} \Sigma \lambda e^{-\kappa\lambda^2 t} \frac{J_m(\lambda r')\, J_m(\lambda r)\, U_m(\lambda a)}{J_m'(\lambda a)},$$

the summation being taken over the positive roots of the equation $J_m(\lambda a) = 0$.

But $$U_m(x)\, J_m'(x) - J_m(x)\, U_m'(x) = \frac{1}{x}.*$$

and therefore the expression for v may be written

$$\frac{2}{\pi a^2} \Sigma a_m \cos m(\theta - \theta') \sum_{\lambda} e^{-\kappa\lambda^2 t} \frac{J_m(\lambda r')\, J_m(\lambda r)}{[J_m'(\lambda a)]^2}.$$

This is the value of the temperature at points (r, θ), $(r > r')$, in the infinite cylinder $r = a$, due to a source at $t = 0$ at the points (r', θ').

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* Cf. Weber. *Über die stationären Strömungen der Electricität in Cylindern.* Crelles' Journal. Bd. 76, p. 10.
Graf u. Gubler, loc. cit. Erstes Heft, pp. 43-45.

Since for the points $r<r'$, we have to interchange r and r' in the above work, as they enter symmetrically, this expression holds for both cases.

The Green's Function for this case is therefore given by the expression

$$\frac{2}{\pi a^2}\sum_{m=0}^{\infty} a_m \cos m(\theta-\theta')\sum_{\lambda} e^{-\kappa\lambda^2(t-\tau)}\frac{J_m(\lambda r')\,J_m(\lambda r)}{[J_m'(\lambda a)]^2}$$

where $a_0=\frac{1}{2}$ and $a_m=1$ ($m\geqq 1$), and the summation takes place over the positive roots of the equation $J_m(\lambda a)=0$.

The solutions of the temperature problems in the Cylinder follow at once.

In particular when the initial temperature is $f(r)$ we obtain the temperature at $(r,\ \theta)$ by integration in the form

$$v=\frac{2}{a^2}\int_0^a r'f(r')\sum e^{-\kappa\lambda^2 t}\frac{J_0(\lambda r)\,J_0(\lambda r')}{[J_0'(\lambda a)]^2}dr',$$

which may be written

$$\frac{2}{a^2}\sum\frac{J_0(\lambda r)}{[J_0'(\lambda a)]^2}e^{-\kappa\lambda^2 t}\int_0^a r'f(r')\,J_0(\lambda r')dr'.$$

and when the initial distribution is $f(r,\ \theta)$ the solution is given by

$$v=\frac{2}{\pi a^2}\int_0^a\int_0^{2\pi} f(r',\theta')r'\left[\sum_{m=0}^{\infty} a_m\cos m(\theta-\theta')\sum_{\lambda} e^{-\kappa\lambda^2 t}\frac{J_m(\lambda r)J_m(\lambda r')}{[J_m'(\lambda a)]^2}\right]dr'd\theta',$$

the summation extending over the positive roots of

$$J_m(\lambda a)=0.$$

If we assume that this series may be integrated term by term we have for the co-efficient of $J_m(\lambda r)\cos m\theta$ the expression

$$\frac{2}{\pi a^2}\frac{e^{-\kappa\lambda^2 t}}{[J_m'(\lambda a)]^2}\int_0^a\int_0^{2\pi} r'f(r',\ \theta')\,J_m(\lambda r')\cos m\theta' dr'd\theta'.$$

These two series correspond with the expansions obtained for the arbitrary functions $f(r)$ and $f(r,\ \theta)$ by the Fourier Method, and occur here as the limiting cases of the expressions obtained for the temperature when t vanishes.*

* Cf. Gray and Mathews, Chapter VI.

§ 8.

Infinite Cylinder $r = a$. Radiation at Boundary into a Medium at Zero.

Starting with the expression for the source in the infinite solid, we transformed it as before into

$$V = \frac{1}{i\pi^2} \Sigma a_m \cos m(\theta - \theta') \int \lambda e^{-\kappa\lambda^2 t} J_m(\lambda r') U_m(\lambda r) d\lambda \qquad (r > r')$$

the integral being taken over the path (P).

We then obtain the supplementary solution

$$V_1 = -\frac{1}{i\pi^2} \Sigma a_m \cos m(\theta - \theta')$$

$$\int \lambda e^{-\kappa\lambda^2 t} \frac{J_m(\lambda r) J_m(\lambda r')(\lambda U'_m(\lambda a) + h U_m(\lambda a))}{\lambda J_m'(\lambda a) + h J_m(\lambda a)} d\lambda,$$

over the same path (P), and we prove that

$$v = V + V_1,$$

which satisfies the Boundary Condition

$$\frac{\partial v}{\partial r} + hv = 0 \text{ at } r = a,$$

also satisfies the Initial Conditions for a source at (r', θ').

The proof follows exactly the same lines as before. We examine

$$\int \lambda J_m(\lambda r) J_m(\lambda r') \frac{\lambda U_m'(\lambda a) + h\, U_m'(\lambda a)}{\lambda\, J_m'(\lambda a) + h\, J_m(\lambda a)} \cdot d\lambda$$

over the path (P) and show that this vanishes, when $r + r' - 2a < 0$, using the fact that the roots of the equation

$$\lambda J_m'(\lambda a) + h J_m(\lambda a) = 0 \, *$$

are real and simple.

The choice of the path (P) then allows us to deduce that

$$\underset{t=0}{\mathrm{Lt}} \int \lambda e^{-\kappa\lambda^2 t} J_m(\lambda r') J_m(\lambda r) \frac{\lambda\, U_m'(\lambda a) + h\, U_m(\lambda a)}{\lambda\, J_m'(\lambda a) + h\, J_m(\lambda a)} d\lambda,$$

vanishes.

This expression for the temperature, involving Contour Integrals, may be reduced to a Double Infinite Series by taking the path (Q),

* Cf. Heine. *Einige Anwendungen der Residuen-Rechnung.* Crelle's Journal, Bd. 89.

as before, instead of the path (P). By this means the co-efficient of $a_m \cos m(\theta - \theta')$ in the expression for v becomes

$$\frac{2}{\pi} \Sigma \lambda e^{-\kappa\lambda^2 t} J_m(\lambda r) J_m(\lambda r') \frac{\lambda U_m'(\lambda a) + h U_m(\lambda a)}{a\lambda J_m''(\lambda a) + (1+ha) J_m'(\lambda a)},$$

the summation extending over the positive roots of the equation

$$\lambda J_m'(\lambda a) + h J_m(\lambda a) = 0,$$

and substituting for $J_m''(\lambda a)$ we obtain for this term the series

$$\frac{2}{\pi a^2} \Sigma \lambda^2 e^{-\kappa\lambda^2 t} \frac{J_m(\lambda r) J_m(\lambda r')}{\left(h^2 + \lambda^2 - \frac{m^2}{a^2}\right)[J_m(\lambda a)]^2}$$

which holds for $r \gtreqless r'$.

We are thus led to the following expression for the temperature at (r, θ) in the cylinder $r = a$, due to the source at (r', θ') at $t = 0$:—

$$v = \frac{2}{\pi a^2} \sum_{m=0}^{\infty} a_m \cos m(\theta - \theta') \sum_{\lambda} \lambda^2 e^{-\kappa\lambda^2 t} \frac{J_m(\lambda r') J_m(\lambda r)}{\left(h^2 + \lambda^2 - \frac{m^2}{a^2}\right)[J_m(\lambda a)]^2}$$

the summation extending over the positive roots of the equation

$$\lambda J_m'(\lambda a) + h J_m(\lambda a) = 0.$$

The results of the general problems with arbitrary initial temperature and arbitrary temperature for the surrounding medium may be at once deduced. In particular when the initial temperature is $f(r)$ and the medium is at zero, the temperature at (r, θ) at time t is given by

$$v = \frac{2}{a^2} \sum_{\lambda} \lambda^2 e^{-\kappa\lambda^2 t} \frac{J_0(\lambda r)}{(h^2 + \lambda^2)[J_0(\lambda a)]^2} \int_0^a r' f(r') J_0(\lambda r') dr';$$

and this gives the expansion of the arbitrary function $f(r)$ in the form,

$$f(r) = \operatorname*{Lt.}_{t=0} \frac{2}{a^2} \sum_{\lambda} \lambda^2 e^{-\kappa\lambda^2 t} \frac{J_0(\lambda r)}{(h^2 + \lambda^2)[J_0(\lambda a)]^2} \int_0^a r' f(r') J_0(\lambda r') dr';$$

while in the case of an arbitrary initial temperature $f(r, \theta)$ we obtain

$$v = \frac{2}{\pi a^2} \int_0^{2\pi} \int_0^a f(r', \theta') r'$$

$$\left[\sum_{m=0}^{\infty} a_m \cos m(\theta - \theta') \sum_{\lambda} \lambda^2 e^{-\kappa\lambda^2 t} \frac{J_m(\lambda r) J_m(\lambda r')}{\left(h^2 + \lambda^2 - \frac{m^2}{a^2}\right)[J_m(\lambda a)]^2}\right] dr' d\theta'.$$

If we assume that we may integrate this Double Series term by term, this expression gives for the co-efficient of the term in

$$J_m(\lambda r)\cos m\theta$$

the value

$$\frac{2\lambda^2 e^{-\kappa\lambda^2 t}}{\pi a^2\left(h^2+\lambda^2-\frac{m^2}{a^2}\right)[J_m(\lambda a)]^2}\int_0^{2\pi}\int_0^a r' f(r', \theta')\, J_m(\lambda r')\cos m\theta' dr' d\theta',$$

and we obtain for the expansion of $f(r', \theta')$ a series which corresponds with the Fourier-Bessel Series obtained by Fourier's method.*

§ 9.

The solution of the three Dimensional Problems discussed in Hobson's paper §§ 5, 6 follow from Green's Functions which may be at once written down. The case of the sphere may be treated as we have done the cylinder, and the problem of a source between two planes meeting at an angle a admits of a corresponding treatment. This latter problem has been discussed for special cases of a by the Method of Images in a Riemann's Space in my paper in the Proceedings of the London Mathematical Society.† The extension to a solid bounded by planes, cylinders, and spheres offers no special difficulty. I propose to return to these questions in a later paper.

* Cf. Gray and Mathews, Chapter VI.

† Proc. Lond. Math. Soc., Vol. XXX., pp. 151-161.

Generalized forms of the Series of Bessel and Legendre.*

By Rev. F. H. JACKSON.

§ 1.

The object of this paper is to investigate certain series and differential equations, generalizations of the series of Bessel and Legendre.

Throughout the paper

$$[n] \text{ denotes } \frac{p^n - 1}{p - 1}$$

reducing, when $p = 1$, to n.

The series discussed are the following:—

$$y_1 = A\left\{x^{[n]} - \frac{[n][n-1]}{[2][2n-1]}p^2x^{[n-2]} + \frac{[n][n-1][n-2][n-3]}{[2][4][2n-1][2n-3]}p^8x^{[n-4]} - \dots\right\} \quad (1)$$

a generalized form of $P_n(x)$,

the relation between coefficients of x in successive terms being

$$A_{r+1} = -A_r p^{2r+1}\frac{[n-2r+1][n-2r+2]}{[2r][2n-2r+1]};$$

$$y_2 = A\left\{x^{[-n-1]} + \frac{[n+1][n+2]}{[2][2n+3]}p^2x^{[-n-3]}\right.$$
$$\left. + \frac{[n+1][n+2][n+3][n+4]}{[2][4][2n+3][2n+5]}p^6x^{[-n-5]} + \dots\right\} \quad (2)$$

a generalized form of $Q_n(x)$,

the relation between successive coefficients being

$$A_{r+1} = A_r p^3\frac{[n+2r-1][n+2r]}{[2r][2n+2r+1]};$$

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* A short paper containing some of the results in this paper was read at the November meeting of the Society; the paper, in its present form, is dated January 1903.

$$y = A\left\{x^{[n]} + \frac{x^{[n+2]}}{[2][2n+2]} + \frac{x^{[n+4]}}{[2][4][2n+2][2n+4]} + \dots\right\} \quad . \quad (3)$$

a generalized form of $J_n(x)$;

$$y = A\left\{1 + \frac{x^{[1]}}{[1][a_1][a_2][a_3]\dots[a_n]} + \frac{x^{[2]}}{[1][2][a_1][a_1+1]\dots[a_n][a_n+1]} + \dots\right\} (4)$$

§ 2.

If

$$y = x^{[n]}$$

$$\frac{dy}{dx} = [n]x^{[n]-1}$$

$$= [n]x^{\nu[n-1]}.$$

Now differentiate, regarding x^ν as the independent variable; denoting the result by $\frac{d^2y}{dx^{(2)}}$,

$$\frac{d^2y}{dx^{(2)}} = \frac{d}{d(x^\nu)}\left\{\frac{dy}{dx}\right\} = [n][n-1]x^{\nu^2[n-2]}.$$

Similarly,

$$\frac{d^3y}{dx^{(3)}} = \frac{d}{d(x^{\nu^2})}\left\{\frac{d^2y}{dx^{(2)}}\right\} = [n][n-1][n-2]x^{\nu^3[n-3]};$$

and generally

$$\frac{d^ry}{dx^{(r)}} = [n][n-1]\dots[n-r+1]x^{\nu^r[n-r]},$$

that is

$$x^{[r]}\frac{d^ry}{dx^{(r)}} = [n][n-1]\dots[n-r+1]x^{[n]}.$$

§ 3.

Denote

$$C_0y + C_1x\frac{dy}{dx} + C_2x^{[2]}\frac{d^2y}{dx^{(2)}} + \dots + C_sx^{[s]}\frac{d^sy}{dx^{(s)}}$$

by $\phi\left[x\frac{dy}{dx}\right]$.

Then if $y = A_1x^{[m_1]} + A_2x^{[m_2]} + \dots + A_rx^{[m_r]} + \dots$

and $\phi[m]$ denote

$$C_0 + C_1[m] + C_2[m][m-1] + \ldots + C_s[m][m-1]\ldots[m-s+1],$$

$$\phi\left[x\frac{dy}{dx}\right] = A_1\phi[m_1]x^{[m_1]} + A_2\phi[m_2]x^{[m_2]} + \ldots + A_r\phi[m_r]x^{[m_r]} + \ldots .$$

Now choose m_1 so as to make $\phi[m_1] = 0$.

Let a, b, c, etc., be roots of $\phi[m_1] = 0$.

Also choose

$$\begin{array}{ll} A_2\phi[m_2] = A_1, & m_2 = m_1 + l, \\ A_3\phi[m_3] = A_2, & m_3 = m_2 + l, \\ \text{etc.} & \text{etc} \end{array}$$

Then, giving m_1 the value a,

$$\phi\left[x\frac{dy}{dx}\right] = A_1x^{[a+l]} + A_2x^{[a+2l]} + A_3x^{[a+3l]} + \ldots$$

$$= A_1\left\{x^{[a+l]} + \frac{x^{[a+2l]}}{\phi[a+l]} + \frac{x^{[a+3l]}}{\phi[a+l]\phi[a+2l]} + \ldots\right\}$$

$$= Ax^{[l]}\left\{x^{p^l[a]} + \frac{x^{p^l[a+l]}}{\phi[a+l]} + \frac{x^{p^l[a+2l]}}{\phi[a+l]\phi[a+2l]} + \ldots\right\};$$

and $$y = A\left\{x^{[a]} + \frac{x^{[a+l]}}{\phi[a+l]} + \frac{x^{[a+2l]}}{\phi[a+l]\phi[a+2l]} + \ldots\right\}.$$

Denoting this series by $F(x)$, we have

$$\phi\left[x\frac{d.F(x)}{dx}\right] = x^{[l]}F(x^{p^l}). \qquad \text{- - - (A)}$$

In the particular case when $p = 1$ this equation becomes

$$\phi\left[x\frac{dy}{dx}\right] = x^l y.$$

§ 4.

The series $\quad y = 1 + \dfrac{x^{[1]}}{[1][a_1][a_2]\ldots[a_n]} + \ldots\ldots$

comes under the preceding form.

If we denote

$$\frac{(p^{a_1+m}-1)(p^{a_2+m}-1)(p^{a_3+m}-1)\ldots(p^{a_n+m}-1)}{(p-1)^n}$$

by $\Pi[a+m]$,

then

$$\Pi[a+m] = A_0 + A_1[m] + A_2[m][m-1] + \ldots + A_n[m][m-1]..[m-n+1],$$

the coefficients A_0, A_1, A_2, etc., being independent of m and given by

$$A_0 = \Pi[a]$$

$$A_1 = \Pi[a+1] - \Pi[a]$$

$$A_2 = \frac{\Pi[a+2] - \frac{p^2-1}{p-1}\Pi[a+1] - p\Pi[a]}{\frac{p^2-1}{p-1} \cdot \frac{p-1}{p-1}}$$

....

....

$$A_r = \frac{\Pi[a+r]}{[r]!} - \frac{\Pi[a+r-1]}{[r-1]![1]!} + p\frac{\Pi[a+r-2]}{[r-2]![2]!} - \ldots + (-1)^r p^{\frac{1}{2}r.r-1}\frac{\Pi[a]}{[r]!},$$

in which $[r]!$ denotes $\dfrac{p^r-1 \,.\, p^{r-1}-1 \,.\, p^{r-2}-1 \ldots p^2-1 \,.\, p-1}{(p-1)^r}$.

We write

$$A_r = \sum_{s=0}^{s=r} (-1)^s p^{\frac{1}{2}s.s-1} \frac{\Pi[a+r-s]}{[r-s]![s]!}. \; *$$

Now take $$\phi\left[x\frac{dy}{dx}\right] \equiv \sum_{r=0}^{r=n} A_r x^{[r+1]} \frac{d^{r+1}y}{dx^{(r+1)}},$$

A_r being $\displaystyle\sum_{s=0}^{s=r} (-1)^s p^{\frac{1}{2}s.s-1} \frac{\Pi[a+r-s]}{[r-s]![s]!}$.

Then if we operate with $\phi\left[x\frac{dy}{dx}\right]$ on a series of the form

$$y = C_1 x^{[m_1]} + C_2 x^{[m_2]} + \ldots\ldots$$

$\phi[m]$ will be $A_0[m] + A_1[m][m-1] + \ldots\ldots$ to $n+1$ terms

$$\equiv [m]\left\{A_0 + A_1[m-1] + \ldots\ldots + A_n[m-1][m-2]\ldots[m-n]\right\}$$

$$\equiv [m]\Pi[a+m-1].$$

$\phi[m_1]$ vanishes for the following values of m_1:

$$0 \quad ,$$
$$1 - a_1,$$
$$1 - a_2,$$
$$\vdots$$
$$1 - a_n.$$

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* Vol. XXVIII., *Proceedings London Mathematical Society*, p. 479.

By taking $m_1 = 0$, $l = 1$, we have

$$\sum_{r=0}^{r=n} A_r x^{[r+1]} \frac{d^{r+1}y}{dx^{(r+1)}} = Ax\left\{1 + \frac{x^{p[1]}}{[1][a_1][a_2]\ldots[a_n]} + \ldots\ldots\right\} \quad - \quad \text{(B)}$$

$$\text{and } y = A\left\{1 + \frac{x^{[1]}}{[1][a_1][a_2]\ldots\ldots[a_n]} + \ldots\ldots\right\},$$

and similar relations for the other values of m_1, viz., for

$$1 - a_1, \; 1 - a_2, \text{ etc.}:$$

$$\sum_{r=0}^{r=n} A_r x^{[r+1]} \frac{d^{r+1}y}{dx^{(r+1)}}$$

$$= Ax^{[2-a_1]}\left\{1 + \frac{x^{p^{2-a_1}[1]}}{[1].[2-a_1][a_2-a_1+1][a_3-a_1+1]\ldots[a_n-a_1+1]}\right.$$

$$+ \frac{x^{p^{2-a_1}[2]}}{[1][2][2-a_1][3-a_1].[a_2-a_1+1][a_2-a_1+2]\ldots[a_n-a_1+2]}$$

$$\left. + \ldots\ldots\right\},$$

$$y = Ax^{[1-a_1]}\left\{1 + \frac{x^{p^{1-a_1}[1]}}{[1][2-a_1][a_2-a_1+1]\ldots[a_n-a_1+1]} + \ldots\ldots\right\},$$

and $n - 1$ similar equations for the values $1 - a_2$, $1 - a_3$, etc.

§ 5.

Two interesting special cases of the equation (B) are obtained by substituting

(1) $\quad a = a_1 = a_2 = a_3 = a_4 = \ldots = a_n$;

(2) $\quad a = a_1 = a_2 + 1 = a_3 + 2 = a_4 + 3 = \ldots = a_n + n - 1$.

The series $F(x)$ is in case (1)

$$A\left\{1 + \frac{x^{[1]}}{[1][a]^n} + \frac{x^{[2]}}{[1][2][a]^n[a+1]^n} + \ldots\ldots\right\} \quad (1)$$

and in case (2)

$$A\left\{1 + \frac{x^{[1]}}{[1][a]_n} + \frac{x^{[2]}}{[1][2][a]_n[a+1]_n} + \ldots\ldots\right\}; \quad (2)$$

the differential equation being

$$\sum_{r=0}^{r=n} A_r x^{[r+1]} \frac{d^{r+1}y}{dx^{(r+1)}} = xF(x^p). \quad - \quad - \quad - \quad \text{(C)}$$

In case (1) $$\mathrm{A}_r \equiv \sum_{s=0}^{s=r}(-1)^s p^{\frac{1}{2}s.\overline{s-1}}\frac{[a+r-s]^n}{[r-s]!\,[s]!};$$

in case (2) $$\mathrm{A}_r \equiv \sum_{s=0}^{s=r}(-1)^s p^{\frac{1}{2}s.\overline{s-1}}\frac{[a+r-s]_n}{[r-s]!\,[s]!},$$

where $[a+r-s]_n \equiv [a+r-s][a+r-s-1][a+r-s-2]\ldots$ to n factors.

In case (2) A_r simplifies, for the $r+1$ terms of the summation are

$$\equiv p^{r(a-n+r)}\frac{\overline{p^n-1}.\overline{p^{n-1}-1}\ldots\overline{p^{n-r+1}-1}}{\overline{p-1}.\overline{p^2-1}\ldots\overline{p^r-1}}[a]_{n-r}\,.$$

The differential equation for series (2) may be written

$$\sum_{r=0}^{r=n} p^{r(a-n+r)}\frac{\overline{p^n-1}.\overline{p^{n-1}-1}\ldots\overline{p^{n-r+1}-1}}{\overline{p-1}.\overline{p^2-1}\ldots\ldots\ldots\overline{p^r-1}}[a]_{n-r}x^{[r+1]}\frac{d^{r+1}y}{dx^{(r+1)}} = x\mathrm{F}(x^p). \quad \text{(D)}$$

§ 6.

Consider the equation

$$px^{[2]}\frac{d^2y}{dx^{(2)}} + \{1-[n]-[-n]\}x\frac{dy}{dx} + [n][-n]y = x^{[2]}\mathrm{F}(x^{p^2}). \quad \text{(E)}$$

The form of the series $\mathrm{F}(x)$ is to be determined.

Assuming
$$y = \mathrm{A}_1 x^{[m_1]} + \mathrm{A}_2 x^{[m_2]} + \ldots\ldots$$
as a possible form of solution,

$$\phi[m] \equiv [n][-n] + \{1-[n]-[-n]\}[m] + p[m][m-1].$$

Now $$p[m][m-1] \equiv [m]\{[m]-1\};$$

$$\therefore\quad \phi[m] \equiv \{[m]-[n]\}\{[m]-[-n]\}.$$

The values of m_1 for which $\phi[m_1]$ vanishes are

$$m_1 = +n \text{ and } m_1 = -n.$$

Also $$\mathrm{A}_{r+1}\phi[m_{r+1}] = \mathrm{A}_r, \text{ and } m_{r+1} = m_r + 2.$$

Therefore, for $m_1 = +n$,

$$\mathrm{A}_{r+1}\{[m_{r+1}]-[n]\}\{[m_{r+1}]-[-n]\} = \mathrm{A}_r,$$

that is $$\mathrm{A}_{r+1}\{[n+2r]-[n]\}\{[n+2r]-[-n]\} = \mathrm{A}_r;$$

$$\therefore\quad \mathrm{A}_{r+1} = \frac{\mathrm{A}_r}{[2r][2n+2r]},$$

and $\quad y = F(x) = A\left\{x^{[n]} + \frac{x^{[n+2]}}{[2][2n+2]} + \frac{x^{[n+4]}}{[2][4][2n+2][2n+4]} + \dots\right\}$

$\quad = J_{[n]}(x).$

We may write the differential equation

$$px^{[2]}\frac{d^2y}{dx^{(2)}} + \left\{1 - [n] - [-n]\right\}x\frac{dy}{dx} + [n][-n]y = x^{[2]}J_{[n]}(x^{p^2}), \qquad \text{(F)}$$

which reduces to Bessel's Equation when $p = 1$.

§ 7.

Consider the expression

$$C_0 y + C_1 x\frac{dy}{dx} + C_2 x^{[2]}\frac{d^2y}{dx^{(2)}} - \frac{d^2y}{dx^{(2)}}, \qquad - \qquad (1)$$

denoted by $\quad \phi\left[x\frac{dy}{dx}\right] - \frac{d^2y}{dx^{(2)}}.$

Then if $\quad y = A_1 x^{[m_1]} + A_2 x^{[m_2]} + \dots + A_r x^{[m_r]} + \dots,$

performing the operations indicated by (1) we have the expression

$$\begin{array}{ll} A_1\phi[m_1]x^{[m_1]} & - A_1 x^{p^2[m_1-2]}[m_1][m_1-1] \\ + A_2\phi[m_2]x^{[m_2]} & - A_2 x^{p^2[m_2-2]}[m_2][m_2-1] \\ + A_3\phi[m_3]x^{[m_3]} & - A_3 x^{p^2[m_3-2]}[m_3][m_3-1] \\ + \quad \dots\dots\dots & - \quad \dots\dots\dots \quad . \end{array}$$

Choose $\quad m_2 = m_1 - 2,$

$\quad m_3 = m_2 - 2,$

etc.,

and $\quad A_{r+1}\phi[m_{r+1}] = A_r[m_r][m_r - 1].$

Write $\quad C_0 = [n][-n-1],$

$\quad C_1 = 1 - [n] - [-n-1],$

$\quad C_2 = p.$

Then $\quad \phi[m_1] \equiv [n][-n-1] + \{1 - [n] - [-n-1]\}[m_1] + p[m_1][m_1 - 1]$

$\quad \equiv \{[m_1] - [n]\}\{[m_1] - [-n-1]\}.$

The values of m_1 which make $\phi[m_1]$ vanish are

$$m_1 = n \text{ and } m_1 = -n-1.$$

Giving m_1 the value n, we have

$$m_{r+1} = n - 2r,$$

and the relation between successive coefficients in the series y is

$$A_{r+1}\{[n-2r]-[n]\}\{[n-2r]-[-n-1]\} = [n-2r+2][n-2r+1]A_r,$$

which reduces to

$$A_{r+1} = -p^{2r+1}\frac{[n-2r+2][n-2r+1]}{[2r][2n-2r+1]}A_r;$$

$$\therefore\ y = A\left\{x^{[n]} - \frac{[n][n-1]}{[2][2n-1]}p^3x^{[n-2]} + \frac{[n][n-1][n-2][n-3]}{[2][4][2n-1][2n-3]}p^8x^{[n-4]} - \dots\right\}$$

$$= P_{[n]}(x)$$

which is a solution of

$$px^{[2]}\frac{d^2y}{dx^{(2)}} - \frac{d^2y}{dx^{(2)}} + \{1-[n]-[-n-1]\}x\frac{dy}{dx} + [n][-n-1]y$$

$$= P'_{[n-2]}(x) - P'_{[n-2]}(x^{p^2}), \quad \text{(G)}$$

$P'_{[n-2]}(x)$ denoting

$$A[n][n-1]\left\{x^{[n-2]} - p^3\frac{[n-2][n-3]}{[2][2n-1]}x^{[n-4]} + \dots\dots\right\}.$$

Similarly, giving m_1 the value $-n-1$, we obtain a series in which the relation between successive coefficients is given by

$$A_{r+1}\{[-n-1-2r]-[n]\}\{[-n-1-2r]-[-n-1]\}$$
$$= [-n-1-2r+2][-n-1-2r+1]A_r;$$

from which $\quad A_{r+1} = \dfrac{A_r.[n+2r][n+2r-1]}{[2r][2n+2r+1]}p^3;$

$$\therefore\ y = A\left\{x^{[-n-1]} + \frac{[n+1][n+2]}{[2][2n+3]}p^3x^{[-n-3]}\right.$$
$$\left. + \frac{[n+1][n+2][n+3][n+4]}{[2][4][2n+3][2n+5]}p^6x^{[-n-5]} + \dots\right\}$$

$$= Q_{[n]}(x).$$

The equation is

$$px^{[2]}\frac{d^2y}{dx^{(2)}} - \frac{d^2y}{dx^{(2)}} + \{1-[-n-1]-[n]\}x\frac{dy}{dx} + [n][-n-1]y$$

$$= Q'_{[n+3]}(x) - Q'_{[n+3]}(x^{p^2}), \quad \text{(H)}$$

$Q'_{[n+3]}(x)$ denoting

$$A[n+1][n+2]\left\{x^{[-n-3]} + \frac{[n+3][n+4]}{[2][2n+3]}p^3x^{[-n-5]} + \dots\dots\right\}.$$

The equations (G) and (H) reduce to Legendre's Equation when $p = 1$.

Second Meeting, 12*th December*, 1902.

Dr THIRD in the Chair.

Mathematical Correspondence.

ROBERT SIMSON, MATTHEW STEWART, JAMES STIRLING.

[*See page* 2.]

On the Uniqueness of Solution of the Linear Differential Equation of the Second Order.

By Dr PEDDIE.

1. In many problems of physics, even in widely different branches of the subject, the relation satisfied by the variables is expressible by means of a linear differential equation of the second order. In general, "initial" conditions have also to be satisfied. If the equation truly represents the physical conditions in, for example, some case of motion, and if no state of instability exists, the solution must be unique. But it is impossible in any case to say with absolute certainty that the representation is strict. The possible error depends on the error which may be made in observation or experiment, and on the number of independent observations or experiments the results of which have been used as the basis of the "law" expressed by the equation. The probable accuracy of any statement as to the non-existence of instability is also dependent on the rigour and extent of the observational or experimental groundwork. The physicist therefore frequently assumes a form of solution which suits his conditions, and does not trouble himself to enquire whether or not other solutions exist if he finds that the one which he has obtained corresponds sufficiently closely to his facts. This procedure is thoroughly justifiable, seeing that he is as sure of the accuracy of his result as he is of the accuracy of his

original equation, while on the other hand a proof of uniqueness may not be easy to obtain. Even if other solutions were found to exist he would be justified in retaining his own and asserting that constraints, whose action did not appear in the differential equation, prevented the manifestation of instability and so prevented the applicability of the other solutions. At least he would be so justified except in the event of such a solution suiting his facts better than his own did. Yet the fact that such a better solution may be found makes the farther investigation of the question desirable, and the proof of the existence of uniqueness adds, when the accuracy of the solution is verified by experiment, to the probability of the accuracy of the differential equation as a description of facts.

2. A well-known example in which uniqueness of solution is proved is furnished by Laplace's Equation in the theory of potential. The proof is obtained by an application of Green's Theorem. Another example occurs in the theory of the conduction of heat (Thomson and Tait's *Treatise on Natural Philosophy*), Green's Theorem being extended so as to apply.

A similar use of Green's Theorem is made by Picard in his *Traité d'Analyse*. He proves that the solution of the equation

$$a\frac{\partial^2 u}{\partial x^2} + 2b\frac{\partial^2 u}{\partial x \partial y} + c\frac{\partial^2 u}{\partial y^2} + 2d\frac{\partial u}{\partial x} + 2e\frac{\partial u}{\partial y} + fu = 0$$

is unique if $b^2 - ac$ be negative while u has a given succession of values along a sufficiently small closed contour in the x, y plane. The coefficients are any continuous functions of x and y, and a and c are alike in sign.

In the case in which f has a sign opposite to that of a and c, the limitation as to smallness of the contour is not required, but the proof is based on the assumption that the coefficients are analytic functions.

Paraf (*Ann. de la Faculté des Sciences de Toulouse*, 1892) gives a proof of the uniqueness of the solution when $ac > b^2$, apart from any assumption as to the nature of the quantities, when f is zero or of opposite sign to a and c.

The object of the present investigation is to obtain a criterion by means of which we can investigate the problem when f is not of opposite sign to a and c. That restriction, however, is replaced by another.

3. We may write Green's Theorem in the extended form

$$\iint\left[a\frac{\partial U}{\partial x}\frac{\partial V}{\partial x}+b_1\frac{\partial U}{\partial x}\frac{\partial V}{\partial y}+b_2\frac{\partial U}{\partial y}\frac{\partial V}{\partial x}+c\frac{\partial U}{\partial y}\frac{\partial V}{\partial y}+p_1V\frac{\partial U}{\partial x}+p_2U\frac{\partial V}{\partial x}+q_1V\frac{\partial U}{\partial y}+q_2U\frac{\partial V}{\partial y}+2rUV\right]dxdy$$

$$=\int U\left[a\frac{\partial V}{\partial x}+b_1\frac{\partial V}{\partial y}+(p_1+p_2)V\right]dy+\int U\left[b_2\frac{\partial V}{\partial x}+c\frac{\partial V}{\partial y}+(q_1+q_2)V\right]dx$$

$$-\iint U\left[\frac{\partial}{\partial x}\left(a\frac{\partial V}{\partial x}\right)+\frac{\partial}{\partial x}\left(b_1\frac{\partial V}{\partial y}\right)+\frac{\partial}{\partial x}(p_1V)+\frac{V}{U}\frac{\partial}{\partial x}(p_2U)+\frac{\partial}{\partial y}\left(b_2\frac{\partial V}{\partial x}\right)+\frac{\partial}{\partial y}\left(c\frac{\partial V}{\partial y}\right)+\frac{\partial}{\partial y}(q_1U)+\frac{V}{U}\frac{\partial}{\partial y}(q_2U)-2rUV\right]dxdy,$$

where the coefficients are any real continuous functions of x and y.

If we take $U=V$, $b_1+b_2=2b$, etc., with the conditions

$$a\frac{\partial^2V}{\partial x^2}+2b\frac{\partial^2V}{\partial x\partial y}+c\frac{\partial^2V}{\partial y^2}+\left(\frac{\partial a}{\partial x}+\frac{\partial b_2}{\partial y}+2p\right)\frac{\partial V}{\partial x}+\left(\frac{\partial b_1}{\partial x}+\frac{\partial c}{\partial y}+2q\right)\frac{\partial V}{\partial y}+2\left(\frac{\partial p}{\partial x}+\frac{\partial q}{\partial y}-r\right)V=0,$$

or say

$$a\frac{\partial^2V}{\partial x^2}+2b\frac{\partial^2V}{\partial x\partial y}+c\frac{\partial^2V}{\partial y^2}+P\frac{\partial V}{\partial x}+Q\frac{\partial V}{\partial y}+2RV=0\,; \qquad (1)$$

and $V=0$, while its first differential coefficients are finite, on given contours over which the line integrals extend, we have

$$\iint\left[a\left(\frac{\partial V}{\partial x}\right)^2+2b\frac{\partial V}{\partial x}\frac{\partial V}{\partial y}+c\left(\frac{\partial V}{\partial y}\right)^2+2pV\frac{\partial V}{\partial x}+2qV\frac{\partial V}{\partial y}+2rV^2\right]dxdy=0. \qquad (2)$$

We may write this as

$$\iint\left[k_1\left(\frac{\partial V}{\partial x}+\lambda V\right)^2+k_2\left(\frac{\partial V}{\partial y}+\mu V\right)^2+k_3\left(\frac{\partial V}{\partial x}+\nu\frac{\partial V}{\partial y}\right)^2+k_4V^2\right]dxdy=0. \qquad (3)$$

Equating coefficients we get

$$a = k_1 + k_3, \quad p = \lambda k_1, \quad 2r = \frac{p^2}{k_1} + \frac{q^2}{k_2} + k_4.$$
$$b = \nu k_3, \quad q = \mu k_2,$$
$$c = k_2 + \nu^2 k_3,$$

If the ks be all of one sign, (3) can only be satisfied by

$$V = 0, \quad \frac{\partial V}{\partial x} = 0, \quad \frac{\partial V}{\partial y} = 0.$$

Therefore, since V may be supposed to be the difference of two functions each of which is assumed, if possible, to satisfy (1), there cannot be more than one solution of the linear equation (1) subject to the condition that the function shall have a given succession of values along given contours.

4. The condition that the ks shall be of one sign necessitates a and c being of one sign, which we may consider to be positive along with those of the ks. And the relations among the coefficients give

$$b^2 = (a - k_1)(c - k_2),$$

while $a \nless k_1$, $c \nless k_2$, so that, as in Picard's cases, we have

$$ac - b^2 \nless 0.$$

5. From the conditions

$$\frac{\partial a}{\partial x} + 2\frac{\partial b}{\partial y} - \frac{\partial b_1}{\partial y} + 2p = P,$$

$$\frac{\partial b_1}{\partial x} + \frac{\partial c}{\partial y} + 2q = Q,$$

$$\frac{\partial p}{\partial x} + \frac{\partial q}{\partial y} - r = R,$$

we find

$$2r = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} - \frac{\partial^2 a}{\partial x^2} - 2\frac{\partial^2 b}{\partial x \partial y} - \frac{\partial^2 c}{\partial y^2} - 2R,$$

so that the value of r is known, and therefore that of

$$2r = \frac{p^2}{a - k_3} + \frac{q^2 k_3}{ck_3 - b^2} + k_4$$

is known.

The quantity r must essentially be positive, and

$$k_3 < a > \frac{b^2}{c}.$$

We see therefore that the solution of

$$a\frac{\partial^2 V}{\partial x^2} + 2b\frac{\partial^2 V}{\partial x \partial y} + c\frac{\partial^2 V}{\partial y^2} + P\frac{\partial V}{\partial x} + Q\frac{\partial V}{\partial y} + 2RV = 0$$

is unique if the contours upon which V is given do not extend to the negative sides of the curves

$$ac - b^2 = 0, \qquad - \quad - \quad - \quad - \quad - \quad - \quad (4)$$

$$2r = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} - \frac{\partial^2 a}{\partial x^2} - 2\frac{\partial^2 b}{\partial x \partial y} - \frac{\partial^2 c}{\partial y^2} - 2R = 0, \qquad - \quad - \quad (5)$$

$$2r - \frac{\left[P - \frac{\partial a}{\partial x} - 2\frac{\partial b}{\partial y} + \frac{\partial b_1}{\partial y}\right]^2}{4(a - k_3)} - \frac{k_3\left[Q - \frac{\partial c}{\partial y} - \frac{\partial b_1}{\partial x}\right]^2}{4(ck_3 - b^2)} = 0. \qquad - \quad (6)$$

6. The two latter conditions become identical when

$$2(r + R) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} - \frac{\partial^2 a}{\partial x^2} - 2\frac{\partial^2 b}{\partial x \partial y} - \frac{\partial^2 c}{\partial y^2} = 0,$$

and the condition then is that R shall be negative—a special case of Paraf's result.

If the first two conditions are satisfied, the problem reduces to that of the possibility of determining b_1 and k_3 so as to satisfy the third. The extreme cases occur when $k_3 = a$ and $k_3 c = b^2$. If $k_3 = a$ we have $k_1 = 0$, $p = 0$, and the third condition becomes

$$a\left[\int (R + r)dy + f(x)\right]^2 \ngtr 2r(ac - b^2).$$

If $k_3 c = b^2$ it becomes

$$c\left[\int (R + r)dx + \phi(y)\right]^2 \ngtr 2r(ac - b^2).$$

The solution is unique when $f(x)$ or $\phi(y)$ can be chosen so as to suit the inequality.

The curve (6) is a boundary flexible within limits, such that, if it be capable of deformation so as to have the given contours entirely

on its positive side, the solution is unique. In determining whether or not sufficient deformation be possible it might be convenient to assign to q an arbitrary value and use (6) in the form

$$2r - \frac{\left[\mathrm{P} - \frac{\partial a}{\partial x} - 2\frac{\partial b}{\partial y} + \psi(y) + \int \frac{\partial}{\partial y}\left(\mathrm{Q} - \frac{\partial c}{\partial y} - 2q\right)dx\right]^2}{4(a - b_3)} - \frac{k_3 q^2}{4(ck_3 - b^2)} = 0,$$

where ψ is arbitrary. But we shall find that it is possible to dispense with (6).

7. It is possible to explore the x, y plane further, even where the value of r is negative. For this purpose an artifice, similarly employed by Picard, may be used. He remarks that, if B and B′ be any two continuous functions of x and y, we have

$$\iint\left[\frac{\partial(\mathrm{BV}^2)}{\partial x} + \frac{\partial(\mathrm{B'V}^2)}{\partial y}\right]dxdy = 0,$$

since V is zero on the contours when we suppose it to represent the difference of two functions satisfying (1). Thus (2) can be written

$$\iint\left[a\left(\frac{\partial \mathrm{V}}{\partial x} + \frac{\mathrm{B}}{a}\mathrm{V}\right)^2 + 2b\frac{\partial \mathrm{V}}{\partial x}\frac{\partial \mathrm{V}}{\partial y} + c\left(\frac{\partial \mathrm{V}}{\partial y} + \frac{\mathrm{B'}}{c}\mathrm{V}\right)^2 + 2p\mathrm{V}\frac{\partial \mathrm{V}}{\partial x} + 2q\mathrm{V}\frac{\partial \mathrm{V}}{\partial y}\right.$$
$$\left. + \left(2r + \frac{\partial \mathrm{B}}{\partial x} + \frac{\partial \mathrm{B'}}{\partial y} - \frac{\mathrm{B'}}{a} - \frac{\mathrm{B'}}{c}\right)\mathrm{V}^2\right]dxdy = 0.$$

Equating coefficients we have

$$\begin{aligned} a &= k_1 + k_3, \\ b &= \nu k_3, \\ c &= k_2 + \nu^2 k_3, \end{aligned} \qquad \begin{aligned} p + \mathrm{B} &= \lambda k_1, \\ q + \mathrm{B'} &= \mu k_2, \end{aligned} \qquad 2r + \frac{\partial \mathrm{B}}{\partial x} + \frac{\partial \mathrm{B'}}{\partial y} = \frac{(p+\mathrm{B})^2}{k_1} + \frac{(q+\mathrm{B'})^2}{k_2} + k_4.$$

We may assume $\mathrm{B} = -p$, $\mathrm{B'} = -q$, and hence we find that *when the contours on which V is given do not lie, in whole or in part, on the negative side of the curve*

$$2(r - \mathrm{R}) = \frac{\partial \mathrm{P}}{\partial x} + \frac{\partial \mathrm{Q}}{\partial y} - \frac{\partial^2 a}{\partial x^2} - 2\frac{\partial^2 b}{\partial x \partial y} - \frac{\partial^2 c}{\partial y^2} - 4\mathrm{R} = 0,$$

the solution is unique. Thus, provided that the value of R do not exceed that of r, we know that the solution is unique *even when R is positive.*

If we choose $p=0$, $B' = -q$, we get

$$\frac{\partial B}{\partial x} = \frac{B^2}{k_1} + k_4 + R - r,$$

and can consider the case where $r - R$ is negative. Let $-A^2/k_1$ be its greatest negative value, and take $k_4 = A^2/k_1 - (R - r)$, k_1 being assumed constant. These conditions give

$$B = A \tan\frac{A}{k_1}(x + a).$$

By suitable choice of a, we have B remaining continuous throughout any strip of the x, y plane, parallel to the y axis, whose breadth is less than $\pi k_1/A$; that is, less than

$$\delta = \frac{\pi(a_0 c_0 - b_0^2)}{c_0 A},$$

where the *least* value of $\frac{ac - b^2}{c}$ is taken, and $\frac{2A^2 c_0}{a_0 c_0 - b_0^2}$ is the greatest positive value of

$$\frac{\partial^2 a}{\partial x^2} + 2\frac{\partial^2 b}{\partial x \partial y} + \frac{\partial^2 c}{\partial y^2} - \frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y} + 4R.$$

If $a_0 < c_0$ we would take $q = 0$, $B = -p$ and so get a_0 instead of c_0 in the denominator of δ.

If we refer to a new set of rectangular axes, the new x-axis being selected parallel to the least breadth of a given closed contour on which V has a given succession of values, we can determine whether or not we can assert that uniqueness of solution exists under the given condition.

Since δ is never zero, the solution of (1) is always unique, as Picard shows, when the contour is sufficiently small.

Another test is given by taking $B = p = 0$, $B' = [\psi(x) - 1]q$. We then get

$$(r - R) + (r + R)\psi(x) = \frac{\left[\psi(x)\int(R + r)dy\right]^2}{k_2} + k_4.$$

Otherwise we may take $B' = q = 0$, $B = [\phi(y) - 1]p$, which gives

$$(r - R) + (r + R)\phi(y) = \frac{\left[\phi(y)\int(R + r)dx\right]^2}{k_2} + k_4.$$

If the disposable quantities can be chosen to satisfy either condition, the solution is unique.

8. If now we return to the equation at the beginning of § 3, and put $U = \alpha V$, where α is positive and not infinite, equation (1) takes the form

$$a\frac{\partial^2 V}{\partial x^2} + 2b\frac{\partial^2 V}{\partial x \partial y} + c\frac{\partial^2 V}{\partial y^2} + \left(\frac{\partial a}{\partial x} + \frac{\partial b_2}{\partial y} + 2p\right)\frac{\partial V}{\partial x} + \left(\frac{\partial b_1}{\partial x} + \frac{\partial c}{\partial y} + 2q\right)\frac{\partial V}{\partial y} + \left(2\frac{\partial p}{\partial x} + 2\frac{\partial q}{\partial y} + \frac{p_2}{\alpha}\frac{\partial \alpha}{\partial x} + \frac{q_2}{\alpha}\frac{\partial \alpha}{\partial y} - 2r\right)V = 0,$$

while (2) becomes

$$a\alpha\left(\frac{\partial V}{\partial x}\right)^2 + 2b\alpha\frac{\partial V}{\partial x}\frac{\partial V}{\partial y} + c\alpha\left(\frac{\partial V}{\partial y}\right)^2 + \left(a\frac{\partial \alpha}{\partial x} + b_2\frac{\partial \alpha}{\partial y} + 2p\alpha\right)V\frac{\partial V}{\partial x} + \left(b_1\frac{\partial \alpha}{\partial x} + c\frac{\partial \alpha}{\partial y} + 2q\alpha\right)V\frac{\partial V}{\partial y} + \left(p_1\frac{\partial \alpha}{\partial x} + q_1\frac{\partial \alpha}{\partial y} + 2r\alpha\right)V^2 = 0.$$

So we have

$$a\alpha = k_1 + k_3, \qquad k_1\lambda = 2p\alpha + a\frac{\partial \alpha}{\partial x} + b_2\frac{\partial \alpha}{\partial y},$$

$$b\alpha = \nu k_3, \qquad\qquad k_1\lambda^2 + k_2\mu^2 + k_4 = 2r\alpha + p_1\frac{\partial \alpha}{\partial x} + q_1\frac{\partial \alpha}{\partial y}.$$

$$c\alpha = k_2 + \nu^2 k_3, \qquad k_2\mu = 2q\alpha + b_1\frac{\partial \alpha}{\partial x} + c\frac{\partial \alpha}{\partial y},$$

Also

$$\frac{\partial a}{\partial x} + \frac{\partial b_2}{\partial y} + 2p = P,$$

$$\frac{\partial b_1}{\partial x} + \frac{\partial c}{\partial y} + 2q = Q,$$

$$2\frac{\partial p}{\partial x} + 2\frac{\partial q}{\partial y} + \frac{p_2}{\alpha}\frac{\partial \alpha}{\partial x} + \frac{q_2}{\alpha}\frac{\partial \alpha}{\partial y} - 2r = 2R.$$

Hence

$$\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} - \frac{\partial^2 a}{\partial x^2} - 2\frac{\partial^2 b}{\partial x \partial y} - \frac{\partial^2 c}{\partial y^2} = 2R + 2r + \frac{p_2}{\alpha}\frac{\partial \alpha}{\partial x} + \frac{q_2}{\alpha}\frac{\partial \alpha}{\partial y}.$$

The remaining condition for uniqueness is that

$$2ra + p_1\frac{\partial a}{\partial x} + q_1\frac{\partial a}{\partial y} = \frac{\left(Pa + a\frac{\partial u}{\partial x} - a\frac{\partial a}{\partial x} + b_2\frac{\partial u}{\partial y} - a\frac{\partial b_2}{\partial y}\right)^2}{k_1} + \frac{\left(Qa + b_1\frac{\partial u}{\partial x} - a\frac{\partial b_1}{\partial x} + c\frac{\partial a}{\partial y} - a\frac{\partial c}{\partial y}\right)^2}{k_2} + k_4.$$

In particular we may take $p_2 = q_2 = b_2 = 0$, in which case the condition becomes

$$2ra + \left(P - \frac{\partial a}{\partial x} - \frac{\partial b}{\partial y}\right)\frac{\partial a}{\partial x} + \left(Q - \frac{\partial b}{\partial x} - \frac{\partial c}{\partial y}\right)\frac{\partial a}{\partial y} = \frac{\left(Pa + a\frac{\partial a}{\partial x} - a\frac{\partial a}{\partial x} + b\frac{\partial a}{\partial y} - a\frac{\partial b}{\partial y}\right)^2}{k_1} + \frac{\left(Qa + b\frac{\partial a}{\partial x} - a\frac{\partial b}{\partial x} + c\frac{\partial a}{\partial y} - a\frac{\partial c}{\partial y}\right)}{k_2} + k_4.$$

By means of this condition we can investigate cases independently of the sign of r.

The special case in which a is constant gives

$$2\frac{r}{a} = \frac{\left(P - \frac{\partial a}{\partial x} - \frac{\partial b}{\partial y}\right)^2}{k_1} + \frac{\left(Q - \frac{\partial b}{\partial x} - \frac{\partial c}{\partial y}\right)^2}{k_2} + k_4.$$

This equation can always be satisfied when r is positive. Thus *the solution of* (1) *is unique when the contours on which V is given lie in the region where*

$$2r = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} - \frac{\partial^2 a}{\partial x^2} - 2\frac{\partial^2 b}{\partial x \partial y} - \frac{\partial^2 c}{\partial y^2} - 2R$$

is positive.

9. We may extend the investigation to regions where r is negative by means of the method used in § 7, a being constant. The equation becomes

$$2ra+\frac{\partial B}{\partial x}+\frac{\partial B'}{\partial y}=\frac{(p+B)^2}{k_1}+\frac{(q+B')}{k_2}+k_4.$$

Putting $p=0$, $B'=-q$, we find

$$\frac{\partial B}{\partial x}=\frac{B^2}{k_1}+k_4-(R+r)+2ra.$$

Taking $k_4=A^2/k_1-(R+r)+2ra$, where A^2/k_1 is the greatest positive value of $R+r(1-2a)$, k_1 being constant, we see that B can be made continuous throughout any strip, parallel to the y axis, whose breadth does not exceed

$$\pi\frac{k_1}{A}.$$

When R is positive and r negative, the value of a which makes this quantity a maximum should be chosen. As the least value of k_2 is b^2/ca, the greatest value of k_1 is $(aca^2-b^2)/ca$. The largest constant value which k_1 can be given cannot exceed the smallest value which $(aca^2-b^2)/ca$ can take when a is fixed.

By putting

$$B=p=0,\ B'=(\psi(x)-1)q,\ \text{or}\ B'=q=0,\ B=(\phi(y)-1)p,$$

respectively, we get

$$2ra+(\psi(x)-1)(R+r)=\frac{[\psi(x)\int(R+r)dy]^2}{k_2}+k_4,$$

$$\text{or}\quad 2ra+(\phi(y)-1)(R+r)=\frac{[\phi(y)\int(R+r)dx]^2}{k_2}+k_4,$$

instead of the conditions given at the end of § 7, the positive constant a being now at disposal in addition to the other quantities.

If we put $B=-p$, $B'=-q$, the condition, with a constant, becomes

$$2r(a-1)-R=k_4.$$

Hence we see that, *even if R be positive and r be negative*, the solution is unique provided that $2r(a-1) > \mathrm{R}$, $a < 1$.

10. There is no limitation on the size of the contours over which V is given, where r and R have the same sign.

By change of the variables an equation may often be put into a form in which one or other of the conditions R negative, or r positive, holds.

The methods above used may be extended to the case of more than two independent variables.

Note on two intrinsically related plane curves.

By R. F. DAVIS, M.A.

The tangent at a point P to a given plane curve intersects another given curve in Q and makes with the tangent at Q to the latter curve a variable angle ψ. It is required to connect the curvatures at P and Q with the length PQ and the angle ψ.

FIGURE 1.

Employing the usual notation and using capital letters for corresponding elements of the outer curve,

$$PT = TP' = \tfrac{1}{2}\delta s;$$

$$Q'N = \tfrac{1}{2}\delta s + (PQ + \delta PQ) - (PQ - \tfrac{1}{2}\delta s)$$

$$= \delta s + \delta PQ;$$

$$\cot\psi = (\delta s + \delta PQ)/PQ \, . \, \delta\phi;$$

$$\therefore \quad PQ\cot\psi = \frac{d}{d\phi}(s + PQ). \qquad \text{(A)}$$

Again $$\delta S = QQ' = (PQ \, . \, \delta\phi)\operatorname{cosec}\psi;$$

$$\therefore \quad PQ\operatorname{cosec}\psi = \frac{dS}{d\phi}. \qquad \text{(B)}$$

Lastly $$\Phi = \phi + \psi. \qquad \text{(C)}$$

These formulæ are sufficient, and will frequently be found useful.

ILLUSTRATIONS.

(1) Let $\psi = \frac{\pi}{2}$, so that QP is always normal at Q to the locus of Q.

(A) gives (when $\cot\psi = 0$) $\frac{d}{d\phi}(s + PQ) = 0$, $\therefore$ $s + PQ =$ constant.

(B) and (C) give $PQ = \frac{dS}{d\Phi}$.

(2) If PQ is constant, $\cot\psi = \rho/\text{PQ}$, and the normal at Q passes through the centre of curvature at P.

Thus, if on the tangent at each point of a curve a constant length measured from the point of contact be taken, then the normal to the locus of the points so found passes through the centre of curvature of the proposed curve. (Bertrand.) *

(3) If PQ and ψ are both constant,

$$\text{R} = \frac{d\text{S}}{d\Phi} = \frac{d\text{S}}{d\phi} = \text{PQ}\operatorname{cosec}\psi\,;$$

$$\therefore \quad \operatorname{cosec}\psi = \text{R}/\text{PQ}\,,$$

and the normal at P passes through the centre of curvature at Q.

Thus, if through each point of a curve a line of given length be drawn making a constant angle with the normal, the normal to the curve locus of the extremities of this line passes through the centre of curvature of the proposed curve. †

(4) *Curve of Pursuit.* If Q describes a straight line with constant velocity v and the velocity of P in the direction of PQ is constant and equal to v/e, then P describes a curve of pursuit.

$$\frac{d\text{S}}{ds} = \frac{d\text{S}}{dt}\Big/\frac{ds}{dt} = e\,; \quad \frac{d\text{S}}{d\phi} = e\rho \text{ and } \frac{d\text{S}}{d\phi} = \text{PQ}\operatorname{cosec}\psi\,;$$

$$\therefore \quad \rho = \frac{\text{PQ}}{e\sin\psi} = \frac{\text{PQ}^2}{e(\text{perp. from P on locus of Q})}\,. \ \ddagger$$

(5) The tangent at a point P on a given curve cuts a given straight line AB in Q. Prove that when PQ is a maximum or a minimum the line through Q perpendicular to AB passes through the centre of curvature at P.

Here Φ is constant, and $d(\text{PQ}) = 0$. Thus $\text{PQ}\cot\psi = \rho$, as in (2).

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* *Williamson's Diff. Calc.*, Chapter XVII., p. 296, No. 27.

† *Bertrand, Diff. Calc.*, quoted by Williamson as above.

‡ *Tait and Steele*, Chapter I., Art. 30 and Ex. 20.

(6) Let P, P′ be two points on an ellipse and Q a point on a confocal ellipse (Fig. 2).

$$\frac{d}{d\phi}(s+\mathrm{PQ})+\frac{d}{d\phi}(s'+\mathrm{P'Q})=\frac{d}{d\phi}(s+\mathrm{PQ})+\frac{d}{d\phi'}(s'+\mathrm{P'Q})\cdot\frac{d\phi'}{d\phi}$$

$$=\mathrm{PQ}\cot\psi+\mathrm{P'Q}\cot\psi\frac{d\phi'}{d\phi}$$

$$=0;$$

for $\dfrac{dS}{d\phi}=\mathrm{PQ}\operatorname{cosec}\psi$ and $-\dfrac{dS}{d\phi'}=\mathrm{P'Q}\operatorname{cosec}\psi$,

$$\therefore\quad \frac{d\phi'}{d\phi}=-\frac{\mathrm{PQ}}{\mathrm{P'Q}}.$$

Integrating, $\qquad s+\mathrm{PQ}+s'+\mathrm{P'Q}=\text{constant}.$

This is Dr Graves' Theorem.

(7) Through each point of a given curve lines are drawn making a constant angle with the normal at that point; show that the normal to their envelope passes through the centre of curvature of the corresponding point on the given curve, and that the projection of the centre of curvature at the evolute of the given curve on this normal is the centre of the curvature of the envelope. *

Here ψ is constant, and $\mathrm{R}=\dfrac{dS}{d\Phi}=\dfrac{dS}{d\phi}=\mathrm{PQ}\operatorname{cosec}\psi$, as in (3).

$$\frac{d\mathrm{R}}{d\Phi}=\frac{d\mathrm{R}}{d\phi}=\operatorname{cosec}\psi\frac{d\mathrm{PQ}}{d\phi}=\operatorname{cosec}\psi(\mathrm{PQ}\cot\psi-\rho)$$

$$\therefore\quad \mathrm{R}\cos\psi-\frac{d\mathrm{R}}{d\Phi}\sin\psi=\rho.\quad \text{(Fig 3.)}$$

(8) A traveller in a railway carriage moving on a curve in a flat country is seated with his face towards the engine and looks out on the inside of the curve. Show that the country beyond a certain point disappears, and that the country nearer than this point comes into view, also, that the point moves with a velocity $\dfrac{d\rho}{dt}\sin\alpha+v\cos\alpha$,

* *Tripos*, 1881.

where ρ is the radius of curvature of the curve, v the velocity of the train and a the angle which the plane from the traveller's eye to the vertical edge nearest the engine of the window which bounds his view, makes with the direction of the train's motion.*

Here ψ is constant, (equal to a).

$$R = PQ\operatorname{cosec}\psi, \text{ as in (3)};$$

or $$\frac{dS}{d\phi} = \sec\psi\left\{\frac{ds}{d\phi} + \frac{d(PQ)}{d\phi}\right\}$$

$$= \sec\psi\left\{\frac{ds}{d\phi} + \frac{d(PQ\operatorname{cosec}\psi)}{d\phi}\sin\psi\right\};$$

or, multiplying by $\frac{d\phi}{dt}$

$$V = \sec\psi\left\{v + \frac{dR}{dt}\sin\psi\right\}.$$

This is for motion *opposite* to that considered in the question. Changing the signs of V and v,

$$-V = \sec\psi\left(-v + \frac{dR}{dt}\sin\psi\right)$$

$$\therefore\quad v = \frac{dR}{dt}\sin\psi + V\cos\psi.$$

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* *Tripos*, 1873 (?) (Dr Pirie).

Third Meeting, 9th January 1903.

Dr Third, President, in the Chair.

Notes on Antireciprocal Points.

By A. G. Burgess, M.A.

Definition. If x, y, z and ξ, η, ζ be the perpendiculars on the sides BC, CA, AB of the $\triangle$ ABC from points O and O′, then O and O′ are antireciprocal points if $x\xi : y\eta : z\zeta :: \tan A : \tan B : \tan C$.

I. Construction to find a point antireciprocal to O (Fig. 4).

Draw through O a line MN antiparallel to BC. Draw OY perpendicular to AC, and OZ perpendicular to AB. Draw lines parallel to AB and AC, and at distances from them respectively equal to YN and MZ, and let them cut in P. Join AP. Find a similar line BQ, and let AP and BQ cut in O′. O′ is the required point. Let the perpendiculars from O be x, y, z and those from O′, ξ, η, ζ.

$$\begin{aligned}\eta : \zeta &= MZ : YN \\ &= OZ/\tan OMZ : OY/\tan ONY \\ &= \tan B/OY : \tan C/OZ \\ &= \tan B/y : \tan C/z\end{aligned}$$

$$\therefore \quad y\eta : z\zeta = \tan B : \tan C.$$

Similarly $$x\xi : z\zeta = \tan A : \tan C$$

$$\therefore \quad x\xi : y\eta : z\zeta = \tan A : \tan B : \tan C.$$

$\therefore$ O′ is the antireciprocal of O.

II. Construction to find a point antireciprocal to itself (Fig. 5).

Draw AD perpendicular to BC, and produce it to meet the semicircle described on BC as diameter in E. Draw lines parallel to AB and CA, and at distances from them respectively equal to BE and CE. Let them cut in P. Join AP. Find a similar line BQ. Let AP and BQ cut in O. O is the required point.

$$\begin{aligned} y^2 : z^2 &= CE^2 : BE^2 = CD : BD \\ &= CD/AD : BD/AD \\ &= AD/BD : AD/CD \\ &= \tan B : \tan C. \end{aligned}$$

Similarly $\quad x^2 : z^2 = \tan A : \tan C$

$$\therefore \quad x^2 : y^2 : z^2 = \tan A : \tan B : \tan C$$

or $\quad x\xi : y\eta : z\zeta = \tan A : \tan B : \tan C$

where $\quad x = \xi,\ y = \eta,\ z = \zeta.$

$\therefore$ O is the required point.

$$x : y : z = \sqrt{\tan A} : \sqrt{\tan B} : \sqrt{\tan C}$$

so that the point whose trilinear coordinates are

$$\sqrt{\tan A},\quad \sqrt{\tan B},\quad \sqrt{\tan C}$$

is the antireciprocal of itself.

The three triangles formed by drawing through this point lines antiparallel to the sides of the $\triangle$ ABC will be equal. The intercepts cut off on the sides by these antiparallels are proportional to

$$\sqrt{\cot A},\quad \sqrt{\cot B},\quad \sqrt{\cot C}.$$

There are four such points, one internal and three external. Their coordinates are given by $\sqrt{\tan A},\ \pm\sqrt{\tan B},\ \pm\sqrt{\tan C}$.

Definition. The antireciprocal of a line is the locus of the antireciprocals of all points in the line.

1. The antireciprocal of a line is a conic passing through the vertices of the triangle.

Let $lx+my+nz=0$ be the equation of the line expressed in trilinear coordinates. Then since $x\xi:y\eta:z\zeta=\tan A:\tan B:\tan C$, the antireciprocal to $lx+my+nz=0$ is

$$\frac{l\tan A}{\xi}+\frac{m\tan B}{\eta}+\frac{n\tan C}{\zeta}=0,$$

or $$\eta\zeta l\tan A+\zeta\xi m\tan B+\xi\eta n\tan C=0. \qquad (1)$$

This represents a conic passing through the vertices of the triangle.

2. The antireciprocal of the circumcircle is the axis of homology of the triangle and its orthic triangle.

$\eta\zeta\sin A+\zeta\xi\sin B+\xi\eta\sin C=0$ is the equation of the circumcircle. Its antireciprocal is $x\cos A+y\cos B+z\cos C=0$, and this is the equation of the said axis of homology.

3. The antireciprocal of a line through a vertex consists of another line through that same vertex, and the opposite side of the triangle.

Let $lx+my=0$ be a line through C.

The equation of its antireciprocal is

$$\eta\zeta l\tan A+\zeta\xi m\tan B=0,$$

or $$\zeta=0(\mathrm{AB}),\ \eta l\tan A+\xi m\tan B=0 \text{ (a line through C)}.$$

The vertex C is the antireciprocal of any point in the opposite side AB.

4. The antireciprocal of a tangent to the circumcircle is a conic touching the line $\Sigma x\cos A=0$ at the antireciprocal of the point of contact of the tangent and the circumcircle.

The condition that $\Sigma lx=0$ touch the circumcircle $\Sigma\eta\zeta\sin A=0$ is that $\Sigma\sqrt{l\sin A}=0$, and the condition that $\Sigma x\cos A=0$ touch the conic $\Sigma\eta\zeta l\tan A=0$ is that $\Sigma\sqrt{l\tan A\,.\cos A}=0$, the same condition.

If x, y, z are the coordinates of the point in which $\Sigma x\cos A = 0$ touches $\Sigma\eta\zeta l\tan A = 0$,

$$\begin{aligned} x:y:z \quad = \quad & l\tan A(\ \ l\sin A - m\sin B - n\sin C) \\ & : m\tan B(-l\sin A + m\sin B - n\sin C) \\ & : n\tan C\,(-l\sin A - m\sin B + n\sin C)\,; \end{aligned}$$

if ξ, η, ζ are the coordinates of the point in which the line $\Sigma lx = 0$ touches the circumcircle

$$\begin{aligned} \xi:\eta:\zeta \quad = \quad & 1/(\ \ l\sin A - m\sin B - n\sin C) \\ & : 1/(-l\sin A + m\sin B - n\sin C) \\ & : 1/(-l\sin A - m\sin B + n\sin C)\,; \end{aligned}$$

and these two are antireciprocals since $x\xi:y\eta:z\zeta = \tan A:\tan B:\tan C$. The equation of the tangent at C to the conic $\Sigma\eta\zeta l\tan A = 0$ is $\eta l\tan A + \xi m\tan B = 0$; or $\eta l\tan A + \xi m\tan B = 0$ is a tangent to a series of conics $\eta\zeta l\tan A + \zeta\xi m\tan B + \xi\eta n\tan C = 0$, where n has different values. But $\zeta(\eta l\tan A + \xi m\tan B) = 0$ is the antireciprocal of $lx + my = 0$, and the antireciprocal of the conic is $lx + my + nz = 0$. These lines are concurrent in a point in AB, no matter what n may be. Hence, if a number of lines are concurrent in a point in a side of a triangle, their antireciprocals have a common tangent at the opposite vertex, namely that part, passing through the vertex, of the antireciprocal of the line joining the point of concurrence of the lines to the opposite vertex.

FIGURE 6.

5. The antireciprocal of the tangent at a vertex to the circumcircle consists partly of the line joining the vertex to the point of concurrence of the opposite side and the line $\Sigma x\cos A = 0$.

The tangent at the vertex C is $x\sin B + y\sin A = 0$. The antireciprocal of this line is $\xi\cos A + \eta\cos B = 0$ and $\zeta = 0$ (CH_3 and BA). Now the lines $\xi\cos A + \eta\cos B = 0$, $\zeta = 0$, $\Sigma x\cos A = 0 (H_2H_3)$ are concurrent. The side DE of the orthic triangle ($x\cos A + y\cos B - z\cos C = 0$) also passes through H_3.

[illegible]	...ntre ...mology.	Axis of homology.	Tangents at vertices to antireciprocal of axis of homology.
[illegible]	[illegible], $\sqrt{\tan A}$ $\sqrt{\tan B}$ $\sqrt{\tan C}$	H_2H_3, $\frac{x}{\sqrt{\tan A}} + \frac{y}{\sqrt{\tan B}} + \frac{z}{\sqrt{\tan C}} = 0$	CH_3, $\frac{x}{\sqrt{\tan A}} + \frac{y}{\sqrt{\tan B}} = 0$ BH_2, $\frac{x}{\sqrt{\tan A}} + \frac{z}{\sqrt{\tan C}} = 0$ AH_1, $\frac{y}{\sqrt{\tan B}} + \frac{z}{\sqrt{\tan C}} = 0$
$\overline{ABC}$, OL_1L_2	L_3, $\sqrt{\tan A}$ $\sqrt{\tan B}$ $-\sqrt{\tan C}$	DH_3, $\frac{x}{\sqrt{\tan A}} + \frac{y}{\sqrt{\tan B}} - \frac{z}{\sqrt{\tan C}} = 0$	AD, $\frac{y}{\sqrt{\tan B}} - \frac{z}{\sqrt{\tan C}} = 0$ BE, $\frac{x}{\sqrt{\tan A}} - \frac{z}{\sqrt{\tan C}} = 0$ CH_3, $\frac{x}{\sqrt{\tan A}} + \frac{y}{\sqrt{\tan B}} = 0$
$\overline{ABC}$, $\overline{OL_1L_3}$	L_2, $\sqrt{\tan A}$ $-\sqrt{\tan B}$ $\sqrt{\tan C}$	FH_2, $\frac{x}{\sqrt{\tan A}} - \frac{y}{\sqrt{\tan B}} + \frac{z}{\sqrt{\tan C}} = 0$	AD, $\frac{y}{\sqrt{\tan B}} - \frac{z}{\sqrt{\tan C}} = 0$ BH_2, $\frac{x}{\sqrt{\tan A}} + \frac{z}{\sqrt{\tan C}} = 0$ CF, $\frac{x}{\sqrt{\tan A}} - \frac{y}{\sqrt{\tan B}} = 0$
$\overline{ABC}$, OL_2L_3	L_1, $-\sqrt{\tan A}$ $\sqrt{\tan B}$ $\sqrt{\tan C}$	EH_1, $-\frac{x}{\sqrt{\tan A}} + \frac{y}{\sqrt{\tan B}} + \frac{z}{\sqrt{\tan C}} = 0$	AH_1, $\frac{y}{\sqrt{\tan B}} + \frac{z}{\sqrt{\tan C}} = 0$ BE, $\frac{x}{\sqrt{\tan A}} - \frac{z}{\sqrt{\tan C}} = 0$ CF, $\frac{x}{\sqrt{\tan A}} - \frac{y}{\sqrt{\tan B}} = 0$

6. If three lines passing through the vertices be concurrent, then their antireciprocals must also be concurrent, for they pass through the antireciprocal of the point of concurrence of the first three.

For example, the lines joining the vertices to the opposite excentres pass through the incentre, and the lines joining the vertices to the antireciprocals of the excentres pass through the antireciprocal of the incentre. The lines joining the vertices to the opposite exsymmedian points pass through the insymmedian point, and the lines joining the vertices to the antireciprocals of the exsymmedian points pass through the antireciprocal of the insymmedian point, *i.e.*, the orthocentre. The three antireciprocals of the exsymmedian points can thus be easily found, for if the points H_1, H_2, H_3 be found, the intersection of BH_2 and AD gives L_1, the antireciprocal of K_1, the exsymmedian point opposite A. The triangles $L_1L_2L_3$ and ABC form the antireciprocal of triangle $K_1K_2K_3$. The two triangles ABC, $L_1L_2L_3$ have a common centre of homology O, and a common axis of homology H_2H_3.

7. If O, L_1, L_2, L_3 be the points $(\sqrt{\tan A},\ \pm\sqrt{\tan B},\ \pm\sqrt{\tan C})$ found as in Construction II., then

$$\text{the line OCL}_3 \text{ is } \frac{x}{\sqrt{\tan A}} - \frac{y}{\sqrt{\tan B}} = 0,$$

$$L_1CL_2 \text{ is } \frac{x}{\sqrt{\tan A}} + \frac{y}{\sqrt{\tan B}} = 0, \text{ etc.}$$

The line $\quad H_2H_3$ is $\dfrac{x}{\sqrt{\tan A}} + \dfrac{y}{\sqrt{\tan B}} + \dfrac{z}{\sqrt{\tan C}} = 0.$

Each of the lines AOL_1 AL_2L_3 is with the side opposite the vertex through which the line passes, its own antireciprocal. The antireciprocal of

$$\frac{x}{\sqrt{\tan A}} + \frac{y}{\sqrt{\tan B}} + \frac{z}{\sqrt{\tan C}} = 0$$

is $$\zeta\eta\sqrt{\tan A} + \xi\zeta\sqrt{\tan B} + \xi\eta\sqrt{\tan C} = 0.$$

The following table shows the connection between the lines.

Triangles.	Centre of homology.	Axis of homology.	Tangents at vertices to antireciprocal of axis of homology.
ABC, $L_1L_2L_3$	O_1, $\sqrt{\tan A}$, $\sqrt{\tan B}$, $\sqrt{\tan C}$	H_2H_3, $\dfrac{x}{\sqrt{\tan A}} + \dfrac{y}{\sqrt{\tan B}} + \dfrac{z}{\sqrt{\tan C}} = 0$	CH_3, $\dfrac{x}{\sqrt{\tan A}} + \dfrac{y}{\sqrt{\tan B}} = 0$
			BH_2, $\dfrac{x}{\sqrt{\tan A}} + \dfrac{z}{\sqrt{\tan C}} = 0$
			AH_1, $\dfrac{y}{\sqrt{\tan B}} + \dfrac{z}{\sqrt{\tan C}} = 0$
$\overline{ABC}$, $\overline{OL_1L_2}$	L_3, $\sqrt{\tan A}$, $\sqrt{\tan B}$, $-\sqrt{\tan C}$	DH_3, $\dfrac{x}{\sqrt{\tan A}} + \dfrac{y}{\sqrt{\tan B}} - \dfrac{z}{\sqrt{\tan C}} = 0$	AD, $\dfrac{y}{\sqrt{\tan B}} - \dfrac{z}{\sqrt{\tan C}} = 0$
			BE, $\dfrac{x}{\sqrt{\tan A}} - \dfrac{z}{\sqrt{\tan C}} = 0$
			CH_3, $\dfrac{x}{\sqrt{\tan A}} + \dfrac{y}{\sqrt{\tan B}} = 0$
$\overline{ABC}$, $\overline{OL_1L_3}$	L_2, $\sqrt{\tan A}$, $-\sqrt{\tan B}$, $\sqrt{\tan C}$	FH_2, $\dfrac{x}{\sqrt{\tan A}} - \dfrac{y}{\sqrt{\tan B}} + \dfrac{z}{\sqrt{\tan C}} = 0$	AD, $\dfrac{y}{\sqrt{\tan B}} - \dfrac{z}{\sqrt{\tan C}} = 0$
			BH_2, $\dfrac{x}{\sqrt{\tan A}} + \dfrac{z}{\sqrt{\tan C}} = 0$
			CF, $\dfrac{x}{\sqrt{\tan A}} - \dfrac{y}{\sqrt{\tan B}} = 0$
$\overline{ABC}$, $\overline{OL_2L_3}$	L_1, $-\sqrt{\tan A}$, $\sqrt{\tan B}$, $\sqrt{\tan C}$	EH_1, $-\dfrac{x}{\sqrt{\tan A}} + \dfrac{y}{\sqrt{\tan B}} + \dfrac{z}{\sqrt{\tan C}} = 0$	AH_1, $\dfrac{y}{\sqrt{\tan B}} + \dfrac{z}{\sqrt{\tan C}} = 0$
			BE, $\dfrac{x}{\sqrt{\tan A}} - \dfrac{z}{\sqrt{\tan C}} = 0$
			CF, $\dfrac{x}{\sqrt{\tan A}} - \dfrac{y}{\sqrt{\tan B}} = 0$

There are four conics, corresponding to the four points, and each of the six lines passing through the vertices is a tangent to two of the conics. Thus each conic touches the other three conics at different vertices.

The line $px+qy+rz=0$ will touch the conic

$$\eta\zeta l\tan A+\xi\zeta m\tan B+\xi\eta n\tan C=0$$

if $$\sqrt{pl\tan A}\pm\sqrt{qm\tan B}\pm\sqrt{rn\tan C}=0.$$

Hence the line $lx+my+nz=0$ will touch its own antireciprocal if $l\sqrt{\tan A}\pm m\sqrt{\tan B}\pm n\sqrt{\tan C}=0$, that is if the line $lx+my+nz=0$ pass through one of the four points

$$(\sqrt{\tan A},\ \pm\sqrt{\tan B},\ \pm\sqrt{\tan C}).$$

Since $$x:y:z=\frac{\tan A}{\xi}:\frac{\tan B}{\eta}:\frac{\tan C}{\zeta}$$

$$\therefore\quad x\xi\tan B-y\eta\tan A=0,\text{ and } x\xi\tan C-z\zeta\tan A=0.$$

Hence (x, y, z) is the point of intersection of the polars of (ξ, η, ζ) with respect to two degenerate conics,

$$x^2\tan B-y^2\tan A=0,\ x^2\tan C-z^2\tan A=0.$$

Since a line corresponds to a conic, and to a point corresponds the intersection of its polars with respect to two fixed conics, this quadric transformation is a Beltrami one, for a discussion of the difference between which and the Hirst transformation see Mr Charles Tweedie's paper read before the Royal Society of Edinburgh on 15th July 1901.

The conics $x^2\tan B-y^2\tan A=0$, etc., break up into the lines

$$\frac{x}{\sqrt{\tan A}}-\frac{y}{\sqrt{\tan B}}=0,\quad \frac{x}{\sqrt{\tan A}}+\frac{y}{\sqrt{\tan B}}=0,\text{ etc.,}$$

or the lines joining the vertices to the points

$$(\sqrt{\tan A},\ \pm\sqrt{\tan B},\ \pm\sqrt{\tan C}).$$

8. The conic $\eta\zeta l\tan A+\xi\zeta m\tan B+\xi\eta n\tan C=0$ will be a rectangular hyperbola, if

$$l\sin A+m\sin B+n\sin C=0.$$

If the line $lx+my+nz=0$ pass through the insymmedian point ($\sin A$, $\sin B$, $\sin C$), then the condition for a rectangular hyperbola is fulfilled. Hence the antireciprocals of all lines passing through the insymmedian point are rectangular hyperbolas. This is otherwise seen; for if the line pass through the insymmedian point, its antireciprocal must pass through the orthocentre, and is therefore a rectangular hyperbola. In particular, the antireciprocal of the line joining the orthocentre and insymmedian point is the rectangular hyperbola passing through the vertices and these two points. Since five points on it are known, it can easily be drawn by Pascal's theorem. (Fig. 7.)

The coordinates of its centre are

$$\frac{\sin(B-C)}{\cos A},\ \frac{\sin(C-A)}{\cos B},\ \frac{\sin(A-B)}{\cos C}.$$

This point lies on the nine-point circle.

The equation of this rectangular hyperbola is

$$\eta\zeta\sin 2A\sin(B-C)+\xi\zeta\sin 2B\sin(C-A)+\xi\eta\sin 2C\sin(A-B)=0.$$

The line joining the orthocentre and insymmedian point is

$$x\cos^2 A\sin(B-C)+y\cos^2 B\sin(C-A)+z\cos^2 C\sin(A-B)=0.$$

On the Singular Points of Plane Curves.

By T. B. Sprague, M.A., LL.D.

In the general equation of a curve that passes through the origin,

$$a_1x + b_1y + a_2x^2 + b_2xy + c_2y^2 + a_3x^3 + b_3x^2y + c_3xy^2 + d_3y^3 + \ldots = 0, \quad (1)$$

which we may write $u_1 + u_2 + u_3 + u_4 + \ldots = 0$, - - - - (2)

$u_1 \equiv a_1x + b_1y = 0$ is the equation to the tangent at the origin; for, when x and y are very small, we may neglect all the terms in comparison with those involving the first powers of x and y. If neither a_1 nor b_1 vanishes, we may, without loss of generality, write the equation in the form

$$y = mx + u_2 + u_3 + u_4 + \quad - \quad - \quad - \quad - \quad (3)$$

Suppose that, when y is put equal to mx, the right hand member becomes $mx + U_2x^2 + U_3x^3 + U_4x^4 + \ldots\ldots.$ Then, putting $y = mx$ in the terms of two dimensions in (3), we have approximately $y = mx + U_2x^2$.

By means of this equation we find the radius of curvature at the origin to be

$$R_0 = \frac{(1 + m^2)^{\frac{3}{2}}}{2U_2}.$$

The curve will, in the neighbourhood of the origin, lie wholly above or wholly below the tangent at the origin, according as U_2 is positive or negative. If this quantity vanishes, R_0 is infinite. In this case $y - mx$ is a factor of u_2. Putting $y = mx$ in equation (3), and neglecting terms that involve higher powers of x than the third, we have now $y = mx + U_3x^3$. Here the term involving x^3 changes sign as x passes through the value zero; and the curve, therefore, no longer lies on the same side of the tangent on both sides of the origin, but there is a point of inflection. The tangent at the origin meets the curve in three coincident points. In this case the normal at the origin is an asymptote to the evolute. (See Fig. 8.)* If we draw a line through the origin, cutting the curve in two neighbour-

* In all the figures the darker lines represent the evolute.

ing points, and the line then moves so as to become the tangent at the origin, those two points move up and coincide with the origin, so that the three coincident points are fully accounted for.

If it should happen, however, that $y-mx$ is a factor also of u_3, we have $U_3=0$; the term involving x^3 vanishes, and we have to take the next higher term. Reasoning as above, we now get the approximate equation to the curve, $y=mx+U_4x^4$. In this case, since U_4x^4 does not change sign as x passes through zero, the curve lies wholly on one side of the tangent in the neighbourhood of the origin; and there is no point of inflection, but we have a point of "undulation". The radius of curvature is still infinite, since U_2 is still zero, and the normal at the origin is still an asymptote to the evolute; but now both branches of the evolute touch the normal at infinity on the same side of the tangent; in fact, there is a cusp at infinity on the evolute. (See Fig. 9.) The tangent at the origin meets the curve in four coincident points; but it is not possible to explain these as in the former case.

If further terms in the expansion of y vanish, and the first that does not vanish when mx is put for y, is the one involving x^r, then there will be a point of inflection or not at the origin, according as r is odd or even. The tangent at the origin will meet the curve in r coincident points; but any other line through the origin, will meet it there in only one point.

It may now be useful to consider the case where R_0 is a maximum or minimum. If $y=mx+Ax^2+Bx^3+Cx^4+\ldots$, we find that

$$R=R_0\left\{1+\left(\frac{6mA}{1+m^2}-\frac{3B}{A}\right)x+\text{ terms involving } x^2,\ x^3, \text{ etc.}\right\} \quad - \quad (4)$$

When x is small, $R-R_0$ changes sign with x; but if the coefficient of x in the above value of R, vanishes, and the coefficient of x^2 remains finite, $R-R_0$ will no longer change sign, and R_0 will be a maximum or minimum value of R. In consequence of this, there will be a cusp in the evolute at the point corresponding to the origin. (See Fig. 10.) The equation (4) shows that the condition for this is

$$B=\frac{2mA^2}{1+m^2}.$$

The origin is in this case a special kind of point; but, as its properties are generally lost on projection, it is not reckoned among the singular points. I propose to call such a point an "apse".

Regarding only the real portions of the curves, the parobola has one apse, and its evolute has one cusp; the hyperbola has two apses, and its evolute has two cusps; and the ellipse has four apses, and its evolute four cusps. If $m=0$, we must have $B=0$ when there is an apse, and the equation becomes $y=Ax^2+Cx^4+\dots$.

If both a_1 and b_1 vanish, so that the equation reduces to

$$u_2+u_3+u_4+\dots=0,$$

the form of the curve in the neighbourhood of the origin is determined by equating to zero the terms of lowest degree; thus, $u_2=0$; and this shows that any line through the origin meets the curve there in two coincident points. We then get two values of y/x, which may be real and unequal, or equal, or imaginary. First, suppose that the values are real and unequal, namely, m and p; then we may, without loss of generality, take the equation to the curve to be $\qquad (y-mx)(y-px)=u_3+u_4+\dots.$

This indicates that there are two tangents at the origin, $y-mx=0$, and $y-px=0$, each of which meets the curve in three coincident points there; in other words, the origin is a double point or "node". Taking the former of the two tangents, and putting $y=mx$ in $\dfrac{u_3}{y-px}$, we get approximately $y=mx+\dfrac{U_3}{m-p}x^2$.

The radius of curvature of this branch at the origin is $\dfrac{m-p}{2U_3}(1+m^2)^{\frac{3}{2}}$. If $m-p=0$ and U_3 is finite, which is a case we shall consider presently, the radius of curvature is zero. If, on the other hand, $U_3=0$, and $m-p$ not $=0$, then the radius of curvature is infinite. In this case, $y-mx$ is a factor of u_3, and the equation to the curve becomes $(y-mx)(y-px)=(y-mx)v_2+u_4+\dots$,

whence $$y=mx+\frac{u_4+\dots\dots}{y-px-v_2},$$

and approximately $$y=mx+\frac{U_4}{m-p}x^3.$$

This shows that the branch has a point of inflection at the origin; and the tangent $y-mx=0$ will therefore meet the origin in three coincident points; but it also cuts the other branch, and therefore

meets the curve in four coincident points at the origin. If $y-px$ also is a factor of u_3, then the second branch also has a point of inflection at the origin.

Figure 11 represents a case where one branch has a point of inflection at the origin, and the other has not; and fig. 12 represents a case where each branch has such a point. Salmon calls these points "flecnode" and "biflecnode", respectively.

If $y-mx$ is also a factor of u_4, the origin will be a point of undulation in the branch; and if $y-px$ also is a factor of u_4, there will be an undulation in each branch. The effect of these singularities on the evolute is evident from what was said above.

Next suppose that the roots of $u_2=0$ are imaginary, so that there are two imaginary tangents at the origin. The equation to the curve cannot then be satisfied by any very small real values of x and y except $x=0$, $y=0$; there are therefore no real points adjacent to the origin, which is thus an isolated point on the curve, and is called a "conjugate" point, or, by Salmon an "acnode". The former of these terms is objectionable, because it puts out of sight the fact that the point must always, like a node, count as two points. I propose to call such a point a "doublet". Any real line through the origin meets the curve in two coincident real points there; but each of the imaginary tangents meets the curve in three coincident real points at the origin.

Lastly, suppose that the roots of $u_2=0$ are equal, so that $m=p$; then we may take the equation of the curve to be $(y-mx)^2=u_3+u_4+\ldots\ldots$; and putting $y=mx$ in u_3, we get approximately $(y-mx)^2=U_3x^3$, and

$$y=mx\pm\sqrt{U_3}\,.\,x^{\frac{3}{2}}.$$

If U_3 is positive, x cannot be negative; and the equation shows that the curve has two branches, each of which touches $y-mx=0$ at the origin. The origin is therefore a cusp: (see Fig. 13). The value of $\dfrac{d^2y}{dx^2}$ at the origin is infinite, unless $U_3=0$; and the radius of curvature is therefore zero, and the evolute of the curve passes through the origin. The tangent at the origin still meets the curve in three coincident points. If U_3 is negative, x cannot be positive; but there is still a cusp at the origin, with its angle pointed in the opposite direction.

If it should happen that $U_3 = 0$, so that $y - mx$ is a factor of u_3, the equation to the curve takes the form

$$(y - mx)^2 = (y - mx)v_3 + u_4 + \ldots$$

whence $$(y - mx - \tfrac{1}{2}v_3)^2 = \tfrac{1}{4}v_3^2 + u_4 + \ldots$$

and $$y = mx + \tfrac{1}{2}v_3 \pm \sqrt{\tfrac{1}{4}v_3^2 + u_4 + \ldots} \qquad (5)$$

or approximately $$y = mx + \tfrac{1}{2}V_2 x^2 \pm \sqrt{\tfrac{1}{4}V_2^2 + U_4} \,.\, x^2$$

$$= mx + (\tfrac{1}{2}V_2 \pm \sqrt{\tfrac{1}{4}V_2^2 + U_4})x^2 \qquad (6)$$

where V_2 is the value of v_3/x^2 when y is put equal to mx. In this case the curve has two branches, which are of finite and different curvatures; and both touch the line $y - mx = 0$ at the origin. This singularity has been called a tacnode. In this case the normal at the origin is a bitangent to the evolute: (see Fig. 14). The tangent at the origin meets the curve in four coincident points; and any other line through the origin meets it in two coincident points. If we take a line, parallel to the tangent, and cutting the curve in four points, and this line moves parallel to itself and becomes the tangent at the origin, then all these four points coincide with the origin, and the four coincident points are thus fully accounted for. In some other cases the full number of coincident points can be similarly accounted for, by means of the real branches of the curve; for instance, in the cases of the inflection, as we have already seen, in the node, and the cusp.

There may be an apse in either or both of the branches. For the sake of simplicity, we will now put $m = 0$, so that the equation

becomes $$y = \tfrac{1}{2}v_2 \pm \sqrt{\tfrac{1}{4}v_2^2 + u_4 + u_5 + \ldots} \qquad (7)$$

Suppose that $$v_2 = Ax^2 + Bxy + Cy^2.$$

Then, putting $y = 0$ in the right-hand member of (7) we get as a first

approximation $$y = \tfrac{1}{2}Ax^2 \pm \sqrt{\tfrac{1}{4}A^2x^4 + a_4x^4}$$

$$= (\tfrac{1}{2}A \pm \sqrt{\tfrac{1}{4}A^2 + a_4})x^2$$

$$= Hx^2, \text{ suppose.}$$

Then, giving y this value in the equation (7), we have for a second approximation

$$y = \tfrac{1}{2}Ax^2 + \tfrac{1}{2}BHx^3 \pm \sqrt{\tfrac{1}{4}A^2x^4 + \tfrac{1}{2}ABHx^5 + a_4x^4 + b_4Hx^5 + a_5x^5}$$

$$= \tfrac{1}{2}Ax^2 + \tfrac{1}{2}BHx^3 \pm \sqrt{(\tfrac{1}{4}A^2 + a_4)x^4 + (\tfrac{1}{2}ABH + b_4H + a_5)x^5}$$

$$= Hx^2 + \left(\tfrac{1}{2}BH \pm \frac{\tfrac{1}{2}ABH + b_4H + a_5}{2\sqrt{\tfrac{1}{4}A^2 + a_4}}\right)x^3.$$

In order that there may be an apse, the co-efficient of x^3 must vanish, or $\quad BH\sqrt{\tfrac{1}{4}A^2 + a_4} \pm (\tfrac{1}{2}ABH + b_4H + a_5) = 0$;

which, on substituting for H its value, becomes

$$\pm(AB + b_4)\sqrt{\tfrac{1}{4}A^2 + a_4} + \tfrac{1}{4}A^2B + a_4B + \tfrac{1}{2}Ab_4 + a_5 = 0.$$

The one or the other of the branches has an apse, according as we take the upper or lower sign. But if

$$AB + b_4 = 0\text{; and } \tfrac{1}{4}A^2B + a_4B + \tfrac{1}{2}Ab_4 + a_5 = 0,$$

the co-efficient of x^3 will vanish, whichever sign is taken, and there is an apse on each branch. The above equations are equivalent to

$$b_4 = -AB,\ a_5 = -a_4B\,;$$

and the equation to the curve becomes

$$y^2 - y(Ax^2 + Bxy + Cy^2) + a_4x^4 - ABx^3y + c_4x^2y^2$$
$$+ d_4xy^3 + e_4y^4 - a_4Bx^5 + b_5x^4y + \dots = 0.$$

If there is one apse, the evolute has one cusp, as shown in Fig. 15; and if there are two apses, the evolute has two cusps, as shown in Fig. 16.

If U_4 is negative and $|U_4| > \tfrac{1}{4}V_2^2$, the value of y given by equation (6) is imaginary except when $x = 0$, $y = 0$; any line through the origin meets the curve there in two coincident points, and the line $y - mx = 0$ meets the curve there in four coincident points:—just as in the case of a tacnode. The curve has thus an isolated point at the origin, but it is not a doublet, and there is a real bitangent at the origin, $y - mx = 0$; or, rather, two coincident tangents, which both touch the imaginary branch at the origin.

We have a particular case of this when u_3 is absent from the equation in consequence of its coefficients vanishing, and u_4 is a negative square, so that the equation takes the form

$$(y - mx)^2 + (px^2 + qxy + ry^2)^2 = u_5 + \dots.$$

If $U_4 = 0$, so that $y - mx$ is a factor of u_4, and $u_4 = (y - mx)v_3$ we have

$$(y - mx - \tfrac{1}{2}v_2 - \tfrac{1}{2}v_3)^2 = \tfrac{1}{4}(v_2 + v_3)^2 + u_5 + \dots$$

and

$$y = mx + \tfrac{1}{2}v_2 + \tfrac{1}{2}v_3 \pm \tfrac{1}{2}(v_2 + v_3)\sqrt{1 + \frac{4u_5}{(v_2 + v_3)^2} + \dots}$$

$$= mx + \tfrac{1}{2}(v_2 + v_3) \pm \tfrac{1}{2}\left\{v_2 + v_3 + \frac{2u_5}{v_2 + v_3} + \dots\right\}.$$

Taking the upper sign, we get

$$y = mx + v_2 + v_3 + \frac{u_5}{v_2 + v_3} + \dots = mx + V_2 x^2 \text{ approximately.}$$

But, taking the lower sign, we have

$$y = mx - \frac{u_5}{v_2 + v_3} - \dots = mx - \frac{U_5}{V_2}x^3, \quad \text{approximately.}$$

This shows that one branch has an inflection at the origin, which may be therefore called a "tacflecnode". (See Fig. 17.) In this case the tangent at the origin meets the curve in five coincident points. The normal touches one branch of the evolute, and is an asymptote to another branch.

If $V_2 = 0$, so that $v_2 = (y - mx)v_1$, we have

$$(y - mx)^2 = (y - mx)^2 v_1 + u_4 + u_5 + \dots$$

and we may neglect $(y - mx)^2 v_1$ in comparison with $(y - mx)^2$, so that

$$y = mx \pm \sqrt{U_4} \,.\, x^2 \text{ approximately.}$$

In this case the two branches have the same curvature, but are turned in opposite directions, as in Fig. 18.

If U_4 and V_2 both $= 0$, so that $u_4 = (y - mx)v_3$, we have

$$(y - mx)^2 = (y - mx)^2 v_1 + (y - mx)v_3 + u_5 + \dots.$$

In this case the method we have followed hitherto is not applicable. In order to get an approximate equation to the curve, we must neglect those terms which are of a higher degree than the others: thus, as already mentioned, we may neglect $(y - mx)^2 v_1$ in comparison with $(y - mx)^2$; but we cannot say which of the remaining terms above written down is of highest degree, and may be neglected, until we know the degree of $(y - mx)$. We may determine this by trial. Assume that $(y - mx)^2$ and $(y - mx)v_3$ are of the same degree;

then $y - mx$ must be of the third degree, and each of the two terms is of the sixth degree: but u_5 is only of the fifth degree, and therefore must not be neglected. Our assumption, that $(y - mx)^2$ and $(y - mx)v_3$ are of the same degree, is therefore inadmissible. Next assume that $(y - mx)^2$ and u_5 are of the same degree; then $y - mx$ is of the degree $2\frac{1}{2}$, and each of these terms is of the degree 5: then $(y - mx)v_3$ is of the degree $5\frac{1}{2}$, and may be neglected in comparison with the other two. Lastly, assume that $(y - mx)v_3$ and u_5 are of the same degree; then $y - mx$ is of the second degree; but this makes $(y - mx)^2$ of the fourth degree, which is lower than the other two, and this assumption is therefore inadmissible.

The only assumption therefore that is admissible is that $(y - mx)^2$ and u_5 are of the same degree; and putting $y = mx$ in the latter, we get

$$y - mx = \pm \sqrt{U_5} \, . \, x^{\frac{5}{2}}.$$

In this case there is a cusp at the origin; but the radius of curvature is infinite, and I propose to call such a point a "flat cusp". The normal at the origin is an asymptote to the evolute; and as the infinite branches lie on the same side of the normal, there is an inflection at infinity in the evolute. (See Fig. 19.)

If $\frac{1}{4}V_2^2 + U_4 = 0$, but V_2, U_4, are not separately $= 0$, the approximate equation (6) becomes $y = mx + \frac{1}{2}V_2x^2$, and both branches have the same curvature at the origin. In order to distinguish between them, we must carry our approximation further. Resuming equation (5) we have

$$y = mx + \tfrac{1}{2}v_2 \pm \sqrt{\tfrac{1}{4}v_2^2 + u_4 + u_5 + \dots}$$

and putting $y = mx + \frac{1}{2}V_2x^2$ in v_2, u_4, u_5,

we get a result of the form

$$y = mx + \tfrac{1}{2}V_2x^2 + Mx^3 + Nx^4 \pm \sqrt{Px^5 + Qx^6 + Sx^7 + Tx^8}.$$

If P is finite and positive, the approximate equation to the curve is

$$y = mx + \tfrac{1}{2}V_2x^2 \pm \sqrt{P} \, . \, x^{\frac{5}{2}}.$$

In this case there is a cusp with both branches on the same side of the tangent. This has been called a ramphoid cusp, as it somewhat resembles the beak of a bird; and I venture to suggest that it should be called a "beak". Salmon calls it a node-cusp.

By applying the formula for R, we find that, as already men-

tioned, the two branches have the same finite radius of curvature at the origin; also that the curvature of one branch increases, and that of the other diminishes, as we recede from the origin.

For, putting y' for $\frac{dy}{dx}$ and y'' for $\frac{d^2y}{dx^2}$,

$$y' = m + V_2 x \pm \tfrac{5}{2}\sqrt{P}\,.\,x^{\frac{3}{2}} + \ldots$$

$$y'' = V_2 \pm \tfrac{15}{4}\sqrt{P}x^{\frac{1}{2}} + \ldots$$

$$(1+y'^2)^{\frac{3}{2}} = \{1 + m^2 + 2mV_2 x \pm 5m\sqrt{P}\,.\,x^{\frac{3}{2}} + \ldots\}^{\frac{3}{2}}$$

$$= (1+m^2)^{\frac{3}{2}}\left\{1 + \frac{3mV_2}{1+m^2}x + \ldots\right\}$$

and
$$R = \frac{(1+y'^2)^{\frac{3}{2}}}{y''} = \frac{\{1+m^2\}^{\frac{3}{2}}}{V_2}\left\{1 \mp \tfrac{15}{4}\frac{\sqrt{P}}{V_2}x^{\frac{1}{2}} + \frac{3mV_2}{1+m^2}x\right\}$$

$$+ \text{ terms involving } x^{\frac{3}{2}},\ x^2, \text{ etc.,}$$

$$= R_0\left(1 \mp \tfrac{15}{4}\frac{\sqrt{P}}{V_2}x^{\frac{1}{2}}\right) \text{ approximately;}$$

where we retain only the lowest power of x, as all the others may be neglected when x is very small. Bearing in mind that the term $\sqrt{P}\,.\,x^{\frac{5}{2}}$ in the value of y, has a double sign, and denoting by R_1, R_2, the radii of curvature in the two branches near the origin, we have approximately

$$R_1 = R_0\left(1 - \tfrac{15}{4}\frac{\sqrt{P}}{V_2}x^{\frac{1}{2}}\right); \quad R_2 = R_0\left(1 + \tfrac{15}{4}\frac{\sqrt{P}}{V_2}x^{\frac{1}{2}}\right).$$

There is thus an inflection in the evolute at the point corresponding to the origin. The form of the curve at the origin and of the evolute at the corresponding point, are shown in Fig. 20.

There is, in general, as Salmon points out, no point of inflection in the evolute. (The same is the case in any envelop, and the evolute is a particular kind of envelop). The above is, therefore, one of the exceptional cases in which the evolute has an inflection. I have hitherto been unable to identify this exceptional case with one of those indicated by Salmon; but I hesitate to suggest that there has been any oversight on his part, as I have found the whole of his reasoning as to the singular points of the evolute, extremely difficult to follow.

If $P = 0$, the approximate equation to the curve takes the form

$$y = mx + \tfrac{1}{2}V_2x^2 + (M \pm \sqrt{Q})x^3$$

and there is now no cusp, but a tacnode at the origin; and as both branches of the curve have the same radius of curvature at the origin, there is a tacnode in the evolute. (See Fig. 21).

If $$M + \sqrt{Q} \text{ or } M - \sqrt{Q} = \frac{m}{1+m^2} \cdot \frac{V_2^2}{2}$$

there is an apse in one branch of the curve, and a cusp in the evolute. (See Fig. 22).

But if Q also $= 0$, the equation takes the form

$$y = mx + \tfrac{1}{2}V_2x^2 + Mx^3 \pm \sqrt{S} \,.\, x^{\frac{7}{2}}.$$

In this case there is again a ramphoid cusp or "beak"; but it differs from the one considered above, inasmuch as the radius of curvature is now a maximum or minimum at the cusp. For we now have

$$R = R_0\left\{1 + \left(\frac{3mV_2}{1+m^2} - \frac{6M}{V_2}\right)x \pm \frac{35}{4}\frac{\sqrt{S}}{V_2}x^{\frac{3}{2}}\right\};$$

and this shows that, unless $M = \dfrac{mV_2^2}{2(1+m^2)}$, so that the co-efficient of x vanishes, R_1 and R_2 are either both greater or both less than R. In this case there is a "beak" in the evolute, as shown in Fig. 23.

If M has this value, R is no longer a maximum or minimum at the cusp, and there is now again an inflection in the evolute, as in Fig. 20.

If a_2, b_2, c_2 all vanish, as well as a_1, b_1, the equation becomes

$$u_3 + u_4 + u_5 + \ldots = 0.$$

Equating u_3 to zero, we have three values of y/x, or the origin is a triple point. The three values may be (1) all real and unequal, or (2) two of them may be equal, or (3) two of them may be imaginary, or (4) all three may be equal. It is unnecessary to investigate cases (1), (2), or (3), as the branches may have the singularities we have investigated above, and no others. The only case that remains is that in which the three values of y/x are equal. We may then take the equation to be

$$(y - mx)^3 = u_4 + u_5 + \ldots.$$

Proceeding in the usual way, we get the approximate equation to the curve

$$y = mx + \sqrt[3]{U_4} \cdot x^{\frac{4}{3}}.$$

There is only one real branch to the curve, and the tangent at the origin meets the curve in four coincident points; while any other line through the origin meets the curve in three coincident points. There is not, however, a doublet at the origin; for the tangent at a doublet is imaginary, the equation in that case being of the form

$$(y - mx)\{(ax + by)^2 + c^2y^2\} + u_4 + u_5 + \ldots = 0.$$

The radius of curvature at the origin vanishes, and the evolute passes through the origin, and has a cusp there. (See Fig. 24.) The origin would be a point of "undulation" if the radius of curvature were infinite; perhaps we may call the present singularity a point of "condensation".

If $U_4 = 0$, so that $y - mx$ is a factor of u_4, then the equation takes the form $$(y - mx)^3 = (y - mx)v_3 + u_5 + u_6 + \ldots$$ and we have to determine by trial which of these terms may be neglected in comparison with the others.

Since $y = mx$ is the tangent at the origin, x and y are of the same degree, and we may neglect u_6 in comparison with u_5, if the latter does not contain the factor $y - mx$. The equation therefore becomes approximately

$$(y - mx)^3 = (y - mx)v_3 + u_5.$$

If now we assume provisionally that u_5 may be neglected in comparison with the other terms, we get $(y - mx)^2 = v_3$; and putting mx for y in v_3 we get $y = mx \pm \sqrt{V_3} \cdot x^{\frac{3}{2}}$. The terms we have kept are of the degree $4\frac{1}{2}$, and it is therefore correct to reject u_5 in comparison with them.

Next assume that $(y - mx)v_3$ may be neglected. This gives $(y - mx)^3 = U_5x^5$. The terms retained are now of the degree 5, but the one neglected is of the degree $4\frac{2}{3}$, and our assumption is therefore inadmissible.

Lastly, assume that $(y-mx)^3$ may be neglected; then we have $y-mx = -\frac{u_5}{v_3}$; and when we put $y=mx$ on the right hand side, $y = mx - \frac{U_5}{V_3}x^2$. This makes the degree of the terms kept, 5; while the degree of the other is 6, and it is therefore rightly neglected.

It does not seem worth while to give the working out of other cases, and I will therefore only give some results. For brevity put $y-mx=Z$. We have seen that when $U_4=0$, or the equation takes the form

$$Z^3 = Zv_3 + u_5 + u_6 + \dots,$$

the curve has three real branches, for which we have approximately at the origin

$$Z = \pm\sqrt{V_3}\,.\,x^{\frac{3}{2}} \text{ or } -\frac{U_5}{V_3}\,.\,x^2. \quad \text{- (See Fig. 25.)}$$

If now $V_3=0$, the equation becomes

$$Z^3 = Z^2v_2 + u_5 + u_6 + \dots,$$

and there is only one real branch to the curve, for which $Z = \sqrt[3]{U_5}\,.\,x^{\frac{5}{3}}$. In this case there is a point of inflection at the origin; but it is not an ordinary inflection, as the radius of curvature is not infinite, but vanishes. I would propose to call such a point a "twist". In this case the evolute passes through the origin, and has an inflection there. (See Fig. 26.)

If $U_5=0$, but $V_3 \neq 0$, we have

$$Z^3 = Z(v_3+v_4) + u_6 + \dots$$

and
$$Z = \pm\sqrt{V_3}\,.\,x^{\frac{3}{2}} \text{ or } -\frac{U_6}{V_3}\,.\,x^3. \quad \text{- (See Fig. 27.)}$$

If V_3 and U_6 both $=0$, the equation becomes

$$Z^3 = Z^2v_2 + Zv_4 + u_6 + \dots$$

and if the roots of $Z^3 - V_2Z^2 - V_4Z - U_6 = 0 \quad - \quad - \quad - \quad \dots (8)$
are z_1, z_2, z_3, and real, the curve has three branches for which

$$Z = z_1x^2,\ z_2x^2,\ z_3x^2, \text{ respectively.}$$

In this case the normal at the origin will touch three branches of the evolute. (See Fig. 28.) If $z_1=z_2=z_3$, so that all three branches of the curve have the same radius of curvature at the origin, the

three points of contact of the normal with the evolute, will coincide; and we shall have what may be called a triple tacnode in the evolute (see Fig. 29), which, if the origin should be an apse in each branch of the curve, will become a triple cusp. (See Fig. 30.)

If $U_6 = 0$, so that one of the roots of the cubic vanishes, we have

$$Z^3 = Z^2 v_2 + Z(v_4 + v_5) + u_7 + \dots$$

and $$Z = z_1 x^2,\ z_2 x^2, \text{ or } -\frac{U_7}{V_4} x^3. \quad \text{- (See Fig. 31.)}$$

If U_7 also $= 0$, we have

$$Z^3 = Z^2 v_2 + Z(v_4 + v_5 + v_6) + u_8 + \dots$$

and the third branch becomes

$$Z = -\frac{U_8}{V_4} . x^4. \quad \text{- - - (See Fig. 32.)}$$

In these last two cases the values of z_1, z_2 are

$$\tfrac{1}{2}\{V_2 \pm \sqrt{V_2^2 + 4V_4}\}.$$

If $V_4 = 0$, but $U_7 \neq 0$, we have

$$Z^3 = Z^2(v_2 + v_3) + Zv_5 + u_7 + \dots$$

and $$Z = V_2 x^2 \text{ or } \pm\sqrt{-\frac{U_7}{V_2}} . x^{\frac{5}{2}}. \quad \text{- (See Fig. 33.)}$$

Going back to equation (8), if $z_1 = z_2 = z_3$, we have

$$(Z - \tfrac{1}{3}V_2 x^2)^3 = U_7 x^7$$

and $$Z = \tfrac{1}{3}V_2 x^2 + \sqrt[3]{U_7} . x^{\frac{7}{3}}. \quad \text{- (See Fig. 34.)}$$

If V_2 and U_6 both $= 0$, we have

$$Z^3 = Z^2 v_3 + Z(v_4 + v_5) + u_7 + \dots$$

and $$Z = \pm\sqrt{V_4} . x^2 \text{ or } -\frac{U_7}{V_4} . x^3. \quad \text{- (See Fig. 35.)}$$

If both V_4 and $U_6 = 0$, we have

$$Z^3 = Z^2(v_2 + v_3) + Zv_5 + u_7 + \dots$$

and $$Z = V_2 x \text{ or } \pm\sqrt{-\frac{U_7}{V_2}} . x^{\frac{5}{2}}. \quad \text{- (See Fig. 36.)}$$

If V_2, V_4, and U_6, all $= 0$, we have

$$Z^3 = Z^2 v_3 + Z^2 v_3 + Zv_5 + u_7 + \dots$$

and $$Z = \sqrt[3]{U_7} . x^{\frac{7}{3}}. \quad \text{- - - - (See Fig. 37.)}$$

If here $U_7 = 0$, we have

$$Z^3 = Z^3 v_1 + Z^2 v_3 + Z(v_5 + v_6) + u_8 + \ldots$$

and $$Z = \pm \sqrt{V_5} \,.\, x^{\frac{5}{2}} \text{ or } -\frac{U_8}{V_5} x^3. \quad - \quad - \quad \text{(See Fig. 38.)}$$

If V_5 also $= 0$, we have

$$Z^3 = Z^3 v_1 + Z^2(v_3 + v_4) + Z v_6 + u_8 + \ldots$$

and $$Z = \sqrt[3]{U_8} \,.\, x^{\frac{8}{3}} \quad - \quad - \quad - \quad - \quad \text{(See Fig. 39.)}$$

Again, supposing the equation to be reduced to

$$u_4 + u_5 + \ldots = 0,$$

the only case which it is necessary to consider, is the one where there are four real branches at the origin, which all have the same tangent; so that the equation may be written

$$Z^4 \equiv (y - mx)^4 = u_5 + u_6 + \ldots .$$

Proceeding in the usual way, we get as a first approximation

$$Z \equiv y - mx = \sqrt[4]{U_5} \,.\, x^{\frac{5}{4}}.$$

This represents a cusp, at which the radius of curvature is zero; so that the evolute passes through the origin, as in Fig. 13.

If $U_5 = 0$, so that

$$Z^4 = Z v_4 + u_6 + \ldots$$

we have $$Z = \sqrt[3]{V_4} \,.\, x^{\frac{4}{3}} \text{ or } -\frac{U_6}{V_4} \,.\, x^2 \quad - \quad - \quad \text{(See Fig. 40.)}$$

If $V_4 = 0$, so that

$$Z^4 = Z^2 v_3 + u_6 + \ldots$$

we have $Z^2 = px^3$ or qx^3, where p, q are the roots of $z^2 - V_3 z - U_6 = 0$,

and $$Z = \sqrt{p} \,.\, x^{\frac{3}{2}} \text{ or } \sqrt{q} \,.\, x^{\frac{3}{2}}. \quad - \quad - \quad \text{(See Fig. 41.)}$$

The curve now has two cusps at the origin, with the same tangent; and there is a tacnode in the evolute.

If $V_3=0$, we have

$$Z^4 = Z^3 v_2 + u_6 + \dots$$

and $$Z = \pm \sqrt[4]{U_6}\,.\,x^{\frac{3}{2}},$$

which represents a common cusp as in Fig. 13.

But if $U_6=0$, we have

$$Z^4 = Z^2 v_3 + Z v_5 + u_7 + \dots$$

and $$Z = \pm \sqrt{V_3}\,.\,x^{\frac{2}{3}},\ \text{or}\ px^2,\ \text{or}\ qx^2, \text{ - (See Fig 42.)}$$

where p, q are the roots of $V_3 z^2 + V_5 z + U_7 = 0$.

If both V_3 and $U_6=0$, we have $Z^4 = Z^3 v_2 + Z v_5 + u_7 + \dots$

and $$Z = \sqrt[4]{U_7}\,.\,x^{\frac{7}{4}}. \text{ - - - - - (See Fig. 43.)}$$

If here $U_7=0$, we have

$$Z^4 = Z^3 v_2 + Z(v_5 + v_6) + u_8 + \dots$$

and $$Z = \sqrt[3]{V_5}\,.\,x^{\frac{5}{3}}\ \text{or}\ -\frac{U_8}{V_5}\,.\,x^2. \text{ - (See Fig. 44.)}$$

But if V_5 also $=0$, we have

$$Z^4 = Z^3 v_2 + Z^2 v_4 + Z v_6 + u_8 + \dots$$

and $$Z = px^2,\ \text{or}\ qx^2,\ \text{or}\ rx^2,\ \text{or}\ sx^2,$$

where p, q, r, s are the roots of $z^4 - V_2 z^3 - V_4 z^2 - V_6 z - U_8 = 0$.

One other kind of singular point may be mentioned, namely a beak (or ramphoid cusp) at which the radius of curvature is infinitely small. We have such a point at the origin when

$$y = mx + ax^{\frac{4}{3}} + bx^{\frac{3}{2}} + \dots \text{ - - (See Fig. 45.)}$$

The singularity in this case is of a much higher order than any we have considered above; for, on clearing the equation of radicals, there are no terms in it of lower degree than the sixth.

In all cases the figures are drawn as if m were $=0$.

On the decimalization of English money, and some simplifications in long division.

By the late J. Hamblin Smith, M.A., Cambridge.

Communicated by J. D. Hamilton Dickson.

[Abstract.]

A method of expressing a sum of money as a decimal of a £ to 3 places has long been known.* When further decimals are necessary, they may be got by the following simple rule, which appears to be new: † multiply the last two decimals found at any stage of the process (after the first 3 have been obtained) by 4, take the digit in the ten's place of the product, append it to the decimals already found as the next decimal, and repeat the process; with the proviso that should the 4-product end with 48, 68, or 88 the digit in the ten's place is to be taken as 5, 7, or 9 respectively.

The origin of the rule is obvious. One farthing being 1/960 of a £ shows the rule to be an application of the expansion of $\left(1-\frac{4}{100}\right)^{-1}$. Thus we might write $n/960$ as $\cdot 00n0n'0n''\ldots$, where $n'=4n$, $n''=4n'$,.... But while this satisfies all our algebraic requirements, the arithmetical necessities are not yet explained by it. Thus if n were 17, the decimal might be written (algebraically) $\cdot 00\overline{17}0\overline{68}0\overline{272}0\overline{1088}0\overline{4352}$, but arithmetically it takes the form ·0177083333..., and a proof is still wanting ‡ to show that the process giving rise to the latter form is deducible from the algebraic result.

Mr Hamblin Smith's rule may be extended to the division of *any* number by 96(97,98...). The rule is in full:—Multiply the last two digits of the quotient found at any stage§ of the process by 4(3,2...), add the digits in the ten's and unit's places of the product to the next two following digits of the dividend, both pairs taken

* See *Proceedings*, Vol. XX., p. 58, 1902.

† „ „ „ XXI., p. 112, 1903.

‡ „ „ „ „ „ „ „

§ To make a beginning we may imagine the quotient to commence with two zeros.

as they stand, take the digit in the ten's place of this sum and append it to the part of the quotient already found, thus extending the quotient by one digit, then repeat the process: with the proviso that when the above sum ends in one of the numbers *under* the same line which the divisor (in the right-hand column) itself is *under*, the ten's place is to be increased by one, viz. :—

								94
							85	95
					66	76	86	96
			47	57	67	77	87	97
		.38	48	58	68	78	88	98
19	29	39	49	59	69	79	89	99

The process might be extended to such numbers as 996, etc.; and also to 101, 102, etc.

On the Decimalization of Money.

By John W. Butters, M.A., B.Sc.

As stated in the preceding paper, the method of expressing, at sight, shillings, pence, and farthings as a decimal of a pound to 3 places has long been known. It is sometimes referred to as the actuaries' rule. According to De Morgan, it occurs for the first time in Kersey's edition of *Wingate's Arithmetic*, 1673 (p. 191). It is also to be found in *Cocker's Decimal Arithmetic*, 1685 (although in a form which is not quite accurate). In some of the earlier books the method of conversion at sight *from* the decimal form is

given, but not *vice versâ*. It is now found in most modern text-books in one form or another.

The method of extension beyond the third decimal place as given in Mr Hamblin Smith's paper is not quite new. The same method is used by De Morgan (*Companion to the British Almanac*, 1841) to find the *nearest* 4th place; a similar method is given by him in 1848 (*Companion to B. A.*) whereby the *actual* 4th and 5th places are obtained; the same method as Mr Hamblin Smith's, but with a different proviso, occurs in *Jackson's Commercial Arithmetic*, 1893. A much simpler method (given in a footnote to my former paper *) is to be found in the forty-ninth edition of a text-book on Arithmetic by Alexander Ingram and Alexander Trotter. This edition bears the date 1871, and I have reason for believing that the method was inserted by Trotter about that year. Mr F. C. Crawford, Edinburgh, informs me that he was taught the method by Trotter in January or February 1868. It occurs also in *Practical Arithmetic for Senior Classes*, by Henry G. C. Smith. To this there is no date.

So far as I know, no proof of the method has been given, and as this is also wanting in Mr Hamblin Smith's paper, it seems desirable to supply the deficiency.

Considered apart from its application to money, the rule may be stated as follows (a special case being taken for simplicity, although the method and proof are perfectly general):—In reducing $\frac{17}{96}$ to a decimal form, if at any stage we multiply the last two digits found (or their excess over 25, 50, or 75) by 4 and increase this product by 1 for each 24 contained in it, we obtain the next two digits.

We thus get successively the following pairs of digits:—

$$17+0=17$$
$$4\times17+2=70$$
$$4(70-50)+3=83$$
$$4(83-75)+1=33$$
$$4(33-25)+1=33 \quad \text{and so on.}$$

We have now to show that $\cdot17708333\ldots=17/96$. From the method of formation it is easily seen that the series may be written in the following form:—

* See *Proceedings*, Vol. XX., p. 58, 1902.

$$\cdot 17 \quad = \quad 17/100$$
$$\cdot 0070 \quad = \quad (4 \times 17 + 2)/100^2$$
$$\cdot 000083 \quad = \quad \{4(4 \times 17 + 2 - 25 \times 2) + 3\}/100^3$$
$$\cdot 00000033 \quad = \quad [4\{4(4 \times 17 + 2 - 25 \times 2) + 3 - 25 \times 3\} + 1]/100^4$$
$$\cdot 0000000033 = (4[4\{4(4 \times 17 + 2 - 25 \times 2) + 3 - 25 \times 3\} + 1 - 25 \times 1] + 1)/100^5$$
$$\&c. \quad = \quad \&c.$$

Hence

$$\begin{aligned}
\cdot 17708333\ldots = \; & 17/100 \\
& + 4 \,.\, 17/100^2 + 2/100^2 \\
& + 4^2 .\, 17/100^3 + 4 \,.\, 2/100^3 - 2/100^2 + 3/100^3 \\
& + 4^3 .\, 17/100^4 + 4^2 .\, 2/100^4 - 4 \,.\, 2/100^3 + 4 \,.\, 3/100^4 - 3/100^3 + 1/100^4 \\
& + 4^4 .\, 17/100^5 + 4^3 .\, 2/100^5 - 4^2 .\, 2/100^4 + 4^2 .\, 3/100^5 - 4 \,.\, 3/100^4 + 4 \,.\, 1/100^5 - 1/100^4 + 1/100^5 \\
& \qquad \&c. \qquad\qquad \&c.
\end{aligned}$$

$$= 17/100 + 4 \,.\, 17/100^2 + 4^2 .\, 17/100^3 + \ldots$$
$$= (17/100)/(1 - 4/100)$$
$$= 17/96$$

It is here assumed that the same number occurs in the second and third columns (2 in this case); also that in the next two columns the same number (3 in this case) occurs; and so on. This is easily seen to be the case when we consider that each addition of 1 makes a 24 into 25; the following subtraction must therefore be of the *same* number of 25's as we have just previously added ones.

The method may be extended quite generally without the need of using a table such as in the preceding paper. To divide by $100 - a$, we multiply the last two digits found (or their excess over $100/a$, $2(100/a)$, $3(100/a)$, etc.) by a and add 1 for each $(100/a - 1)$ in the product; this gives us the next two digits; and so on.

With $1000 - a$ we use (and get) 3 digits at a time; and so on.

The treatment of such numbers as 104 will be readily seen from a particular case, *e.g.*, 17/104 we get the following pairs of numbers:

$$
\begin{aligned}
17 - 1 &= 16 \\
4(\ 25 - 16) - 2 &= 34 \\
4(\ 50 - 34) - 3 &= 61 \\
4(\ 75 - 61) - 3 &= 53 \\
4(\ 75 - 53) - 4 &= 84 \\
4(100 - 84) - 3 &= 61 \\
\therefore\quad 17/104 &= \cdot 163\dot{4}6153\dot{8}.
\end{aligned}
$$

In this case, instead of multiplying the *excess* over 25, 50, 75, we multiply the *defect* from 25, 50, 75, 100, and instead of *adding* 1 for each 25 that we are about to use in the following step, we *deduct* 1.

It is worthy of note that (as stated in the rule) the process may be applied at *any* stage, *e.g.*, having obtained, in the last example, 16 and then 34, we may continue with 63 instead of with 34.

Fourth Meeting, 13th February, 1903.

Dr MACKAY in the Chair.

Triangles in Multiple Perspective, viewed in connection with Determinants of the third order.

By J. A. THIRD, M.A., D.Sc.

I. *Introductory.*

The theorems given in the present section are fundamental in the theory of triangles in multiple perspective. They are all perfectly well known, but are given here because without them the succeeding sections would be unintelligible.

Two triangles $A_1A_2A_3$, $B_1B_2B_3$ can be in perspective in six different ways, indicated by the following symbols, in which the A's are to be understood as connecting collinearly with the B's standing directly underneath.

$$(1)\ \frac{A_1A_2A_3}{B_1B_2B_3}; \quad (2)\ \frac{A_1A_2A_3}{B_2B_3B_1}; \quad (3)\ \frac{A_1A_2A_3}{B_3B_1B_2};$$

$$(4)\ \frac{A_1A_2A_3}{B_1B_3B_2}; \quad (5)\ \frac{A_1A_2A_3}{B_2B_1B_3}; \quad (6)\ \frac{A_1A_2A_3}{B_3B_2B_1}.$$

If the coordinates of B_1, B_2, B_3, with reference to the triangle A, be given by the first, second, and third rows of the determinant

$$\begin{vmatrix} b_{11}, & b_{12}, & b_{13}, \\ b_{21}, & b_{22}, & b_{23}, \\ b_{31}, & b_{32}, & b_{33}, \end{vmatrix} \equiv \Delta b,$$

the necessary and sufficient conditions that the various modes of perspective enumerated above may obtain, are:

for (1) $b_{21}b_{32}b_{13} = b_{31}b_{12}b_{23}$; for (4) $b_{21}b_{12}b_{33} = b_{31}b_{22}b_{13}$;

for (2) $b_{31}b_{12}b_{23} = b_{11}b_{22}b_{33}$; for (5) $b_{31}b_{22}b_{13} = b_{11}b_{32}b_{23}$;

for (3) $b_{11}b_{22}b_{33} = b_{21}b_{32}b_{13}$; for (6) $b_{11}b_{32}b_{23} = b_{21}b_{12}b_{33}$.

If we say that a positive (or negative) term in the expansion of a determinant of the third order is the first, second, or third positive (or negative) term, according as it contains the first, second, or third element of the first column, we may express the foregoing conditions as follows. We have perspective of the first kind between the triangles A and B when the second and third positive terms of Δb are equal, of the second kind when the third and first positive terms are equal, of the third kind when the first and second positive terms are equal, of the fourth kind when the second and third negative terms are equal, of the fifth kind when the third and first negative terms are equal, and of the sixth kind when the first and second negative terms are equal.

If the two triangles are in perspective in any two out of the first three ways, they are also in perspective in the remaining way, *i.e.*, they are in *triple perspective*. This kind of triple perspective may be called *direct*. If the two triangles are in perspective in any two out of the last three ways, they are also in perspective in the remaining way, *i.e.*, they are as before in triple perspective. This kind of triple perspective may be called *perverse*. It is to be noted that there is no essential difference between direct and perverse triple perspective, it being merely a matter of how the vertices of the triangles are named whether the triple perspective be of the one kind or the other.

When any one of the first three modes of perspective occurs simultaneously with any one of the last three modes, we have *double* perspective. Two triangles may be in double perspective in nine not essentially different ways. When two triangles are in double perspective it may be noted that a certain vertex of the one is twice connected with a certain vertex of the other, and that the two centres of perspective lie on the connector of these vertices.

When the first three conditions hold good simultaneously with any one of the last three, or the last three with any one of the first three, we have *quadruple* perspective. Two triangles may obviously be in quadruple perspective in six ways.

When any five of the conditions hold good simultaneously, they all hold good, and thus the two triangles are in *sextuple* perspective. Two real triangles can be in single, double, triple, or quadruple perspective, but in the case of sextuple perspective, one of the two

triangles must reduce to a point or else be imaginary, *i.e.*, have imaginary points for two of its vertices. If b_1, b_2, b_3 be the coordinates, with reference to a triangle A, of one of the vertices of a triangle B supposed to be in sextuple perspective with A, the other vertices of B either coincide with the point (b_1, b_2, b_3) or are the points $(b_1, \omega b_2, \omega^2 b_3)$ and $(b_1, \omega^2 b_2, \omega b_3)$, where ω and ω^2 are the imaginary cube roots of unity. It is clear that the side of B containing the imaginary vertices is real.

The coordinates used in the conditions given for the various kinds of perspective have been trilinear or areal, but it is important to observe that precisely similar results are obtained when tangential coordinates are used.

II.

I now proceed to the problem of finding the conditions for perspective between two triangles A and B the coordinates of whose vertices are given with reference to a third triangle C.

Let the coordinates of A_1, A_2, A_3 be given by the rows of the determinant

$$\begin{vmatrix} a_{11}, & a_{12}, & a_{13} \\ a_{21}, & a_{22}, & a_{23} \\ a_{31}, & a_{32}, & a_{33} \end{vmatrix} \equiv \Delta a,$$

and those of B_1, B_2, B_3 by the rows of the determinant

$$\begin{vmatrix} b_{11}, & b_{12}, & b_{13} \\ b_{21}, & b_{22}, & b_{23} \\ b_{31}, & b_{32}, & b_{33} \end{vmatrix} \equiv \Delta b,$$

Let B_{ik} be the complementary minor of the element b_{ik} of Δb. Then the equations of the sides of B are

$$B_2B_3 \equiv x_1B_{11} + x_2B_{12} + x_3B_{13} = 0,$$
$$B_3B_1 \equiv x_1B_{21} + x_2B_{22} + x_3B_{23} = 0,$$
$$B_1B_2 \equiv x_1B_{31} + x_2B_{32} + x_3B_{33} = 0.$$

Hence the perpendiculars from A_1, A_2, A_3 to B_2B_3 are proportional to

$$a_{11}B_{11} + a_{12}B_{12} + a_{13}B_{13},\ a_{21}B_{11} + a_{22}B_{12} + a_{23}B_{13},\ a_{31}B_{11} + a_{32}B_{12} + a_{33}B_{13},$$

say to β_{11}, β_{12}, β_{13}.

Similarly the perpendiculars from A_1, A_2, A_3 to B_3B_1 are proportional to

$$a_{11}B_{21}+a_{12}B_{22}+a_{13}B_{23},\ a_{21}B_{21}+a_{22}B_{22}+a_{23}B_{23},\ a_{31}B_{21}+a_{32}B_{22}+a_{33}B_{23},$$

say to $\beta_{21},\ \beta_{22},\ \beta_{23}$.

Similarly the perpendiculars from A_1, A_2, A_3 to B_1B_2 are proportional to

$$a_{11}B_{31}+a_{12}B_{32}+a_{13}B_{33},\ a_{21}B_{31}+a_{22}B_{32}+a_{23}B_{33},\ a_{31}B_{31}+a_{32}B_{32}+a_{33}B_{33},$$

say to $\beta_{31},\ \beta_{32},\ \beta_{33}$.

Hence the tangential coordinates of the sides of B, *with reference to A*, are given by the rows of the determinant

$$\begin{vmatrix} \beta_{11}, & \beta_{12}, & \beta_{13} \\ \beta_{21}, & \beta_{22}, & \beta_{23} \\ \beta_{31} & \beta_{32}, & \beta_{33} \end{vmatrix} \equiv \Delta\beta.$$

It follows from what was said in section I. that A and B are simply in perspective in the first, second, or third way, according as the second and third, third and first, or first and second positive terms of $\Delta\beta$ are equal, and that they are simply in perspective in the fourth, fifth, or sixth way according as the second and third, third and first, or first and second negative terms of $\Delta\beta$ are equal. The conditions for the various kinds of perspective are, in fact, exactly the same as those given in section I., except that for b we have throughout β. Thus, for example, the necessary and sufficient condition that A and B be directly triply in perspective is that the three positive terms of $\Delta\beta$ be equal, and the condition that they be perversely triply in perspective is that the three negative terms of $\Delta\beta$ be equal.

It will be noticed that

$$\Delta\beta \equiv \Delta a \,.\, \Delta' b,$$

where $\Delta' b$ is the adjugate of Δb. In order that the elements of the rows of $\Delta\beta$ may be the tangential coordinates of the sides of B with reference to A, the multiplication of Δa and $\Delta' b$ must be by rows, and, further, must be effected in such a manner that, in the extended array of $\Delta\beta$, the a's in each vertical column may have precisely the same suffixes. It must be observed also that no factors may be removed from the columns of Δa or $\Delta' b$ before multiplica-

tion, unless the same factor be removed from each, otherwise the results would be vitiated. If we removed, for example, a factor from the first column of Δa, the elements of its rows would no longer represent the coordinates of the vertices of A.

A determinant similar to $\Delta\beta$, viz. $\Delta\alpha$, the elements of whose rows are the tangential coordinates of the sides of A *with reference to B*, is obtained by multiplying in exactly the same way Δb and $\Delta' a$.

The conditions that A and B may be in perspective may, of course, be obtained, in another form, by writing down the conditions that the lines joining corresponding vertices may be concurrent. This is the method adopted by Rosanes, who seems to have been the first to deal with the subject of multiple perspective, in his paper, *Über Dreiecke in perspectivischer Lage*, *Math. Ann.*, Bd. II., p. 549.

The condition found in this way for the first mode of perspective

$$\begin{matrix} A_1A_2A_3 \\ B_1B_2B_3, \end{matrix}$$

$$\text{is}\quad \Delta_1 \equiv \begin{vmatrix} a_{12}b_{13} - a_{13}b_{12}, & a_{13}b_{11} - a_{11}b_{13}, & a_{11}b_{12} - a_{12}b_{11} \\ a_{22}b_{23} - a_{23}b_{22}, & a_{23}b_{21} - a_{21}b_{23}, & a_{21}b_{22} - a_{22}b_{21} \\ a_{32}b_{33} - a_{33}b_{32}, & a_{33}b_{31} - a_{31}b_{33}, & a_{31}b_{32} - a_{32}b_{31} \end{vmatrix} \equiv 0.$$

For $\begin{matrix} A_1A_2A_3 \\ B_2B_3B_1 \end{matrix}$ it is $\Delta_2 \equiv 0$, and for $\begin{matrix} A_1A_2A_3 \\ B_3B_1B_2 \end{matrix}$, $\Delta_3 \equiv 0$, where Δ_2 is derived from Δ_1, and Δ_3 from Δ_2 by cyclical interchange of the first suffixes of the b's. Again for

$$\begin{matrix} A_1A_2A_3 \\ B_1B_3B_2 \end{matrix}, \quad \begin{matrix} A_1A_2A_3 \\ B_2B_1B_3 \end{matrix}, \quad \begin{matrix} A_1A_2A_3 \\ B_3B_2B_1 \end{matrix},$$

the conditions are $\Delta_4 \equiv 0$, $\Delta_5 \equiv 0$, $\Delta_6 \equiv 0$ respectively, where Δ_4, Δ_5, Δ_6 are derived from Δ_1, Δ_2, Δ_3 respectively by causing the b's in the second rows of the latter to exchange suffixes with the b's directly underneath in the third rows.

These conditions must of course be equivalent to those already found, and as a matter of fact

$$\Delta_1 . \Delta b \equiv \beta_{21}\beta_{32}\beta_{13} - \beta_{31}\beta_{12}\beta_{23},$$
$$\Delta_2 . \Delta b \equiv \beta_{31}\beta_{12}\beta_{23} - \beta_{11}\beta_{22}\beta_{33},$$
$$\Delta_3 . \Delta b \equiv \beta_{11}\beta_{22}\beta_{33} - \beta_{21}\beta_{32}\beta_{13},$$
$$\Delta_4 . \Delta b \equiv \beta_{21}\beta_{12}\beta_{33} - \beta_{31}\beta_{22}\beta_{13},$$
$$\Delta_5 . \Delta b \equiv \beta_{31}\beta_{22}\beta_{13} - \beta_{11}\beta_{32}\beta_{23},$$
$$\Delta_6 . \Delta b \equiv \beta_{11}\beta_{32}\beta_{23} - \beta_{21}\beta_{12}\beta_{33}.$$

From the above we deduce the following identities connecting the elements of any two determinants, Δa and Δb, of the third order,

$$\Delta_1 + \Delta_2 + \Delta_3 \equiv 0, \text{ and } \Delta_4 + \Delta_5 + \Delta_6 \equiv 0.$$

It may also be worth stating that if $\Delta_4 \equiv \Delta_5 \equiv \Delta_6$, the determinants Δa and $\Delta \beta$ either are circulants or can be reduced to circulants by the removal of common factors from their columns, and that if $\Delta_1 \equiv \Delta_2 \equiv \Delta_3$, the same determinants, when any two of their rows are interchanged, either are circulants or can be reduced to circulants by the removal of common factors from their columns.

III.

If $C_1C_2C_3$ be taken as the fundamental triangle, it is easy to show that all the triangles which are in direct triple perspective with C can have the coordinates of their vertices (or sides) thrown into the form

$$\begin{vmatrix} p_1q_1, & p_2q_2, & p_3q_3 \\ p_1q_2, & p_2q_3, & p_3q_1 \\ p_1q_3, & p_2q_1, & p_3q_2 \end{vmatrix}.$$

I have already made use of coordinates of this form in the paper on "Triangles triply in perspective" which I communicated to the Society two years ago, and more recently they have been employed by Mr Ferrari in a paper "Sur les triangles trihomologiques" in *Mathesis*, Jan. 1902, in the course of which he traverses some of the ground covered by my paper and adds some new results. When the triple perspective is perverse, the second and third rows in the above array must be interchanged.

The foregoing coordinates, as Mr Ferrari has pointed out, may be regarded as the result of multiplying the coordinates of a point

$$P(p_1, \quad p_2, \quad p_3)$$

by those of a triangle Q, viz.,

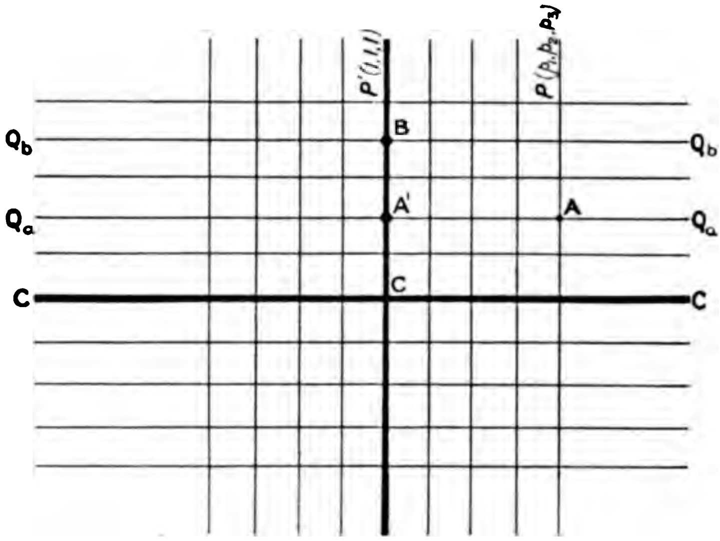
$$\begin{array}{c} Q_1 \\ Q_2 \\ Q_3 \end{array} \left| \begin{array}{ccc} q_1, & q_2, & q_3 \\ q_2, & q_3, & q_1 \\ q_3, & q_1, & q_2 \end{array} \right| .$$

When one of the vertices of Q is given the others are also given, and thus, with respect to the same fundamental triangle C, there are ∞^2 such triangles in the plane. As P can also assume ∞^2 positions, there are ∞^4 triangles in direct triple perspective with C. These triangles are also in perverse triple perspective with C if we take their vertices in a different order. Thus in all there are ∞^4 triangles triply in perspective with C.

If we regard P as fixed and Q as variable, we obtain a series of ∞^2 triangles which, adopting Mr Ferrari's notation, I shall call a T*p* series. Every T*p* series obviously includes the triangle $C_1C_3C_2$. The triangles Q form a T*p* series, P being in this case the point (1, 1, 1). Following the nomenclature of my previous paper, I shall call this particular series the *isobaric* T*p* series, although the name is not strictly appropriate unless the coordinates used be barycentric. The fundamental triangle C and any other triangle in triple perspective with it are sufficient to determine a T*p* series, all the other triangles of the series, such as the triangles formed by their centres and axes of perspective, being derivable from the original two by purely linear constructions. Any T*p* series can be projected into the isobaric T*p* series by projecting P into the point (1, 1, 1), and in this way I proved, in the paper already referred to, a considerable number of theorems respecting the triangles of such a series, *inter alia* that any two of them are in perverse triple perspective with each other.

If we regard Q as fixed and P as variable, we obtain a series of ∞^2 triangles which Mr Ferrari has called a T*q* series. The fundamental triangle C obviously belongs to only one T*q* series, namely, that in which Q coincides with $C_1C_3C_2$. In this case all the triangles of the series coincide with $C_1C_3C_2$.

The ∞^4 triangles in triple perspective with C may be usefully represented by the subjoined diagram consisting of vertical and horizontal lines.



Each vertical line represents a Tp series, the thick vertical line representing the isobaric Tp series, for which P is the point (1, 1, 1). Each horizontal line represents a Tq series, the thick horizontal line representing the Tq series every triangle of which coincides with $C_1C_3C_2$. The ∞^4 triangles are then represented by the points of intersection of the lines, each line being supposed to have ∞^2 points. Every point on the thick horizontal line represents the same triangle $C_1C_3C_2$. Every point on the thick vertical line represents the defining triangle of the Tq series represented by the horizontal line passing through the point.

As has been said, all the triangles of any Tp series are triply in perspective with each other in pairs. The question now naturally arises whether any of the triangles of a Tq series are triply in perspective with each other. This question will be treated in connection with the more general question whether triangles which

belong to different Tp and at the same time different Tq series can be triply in perspective with each other. Another form of this question is: Can two triangles A and B which are triply in perspective with a third triangle C, be triply in perspective with each other without belonging to the same Tp series with respect to C? Two cases fall to be considered, (1) when A, B, and C are supposed to be triply in perspective with each other in the same way, say directly, and (2) when two pairs are triply in perspective in one way, say A, C and B, C directly, and the remaining pair in a different way, say A, B perversely.

C being the fundamental triangle, let A (see diagram) belong to the Tp system defined by the point P(p_1, p_2, p_3), and to the Tq system defined by the triangle Qa, the coordinates of whose vertices are given by the rows of the determinant

$$\begin{vmatrix} a_1, & a_2, & a_3 \\ a_2, & a_3, & a_1 \\ a_3, & a_1, & a_2 \end{vmatrix};$$

and that B belongs to the Tp series defined by the point P′(1, 1, 1), and to the Tq series defined by the triangle Qb, the coordinates of whose vertices are given by the rows of the determinant

$$\begin{vmatrix} b_1, & b_2, & b_3 \\ b_2, & b_3, & b_1 \\ b_3, & b_1, & b_2 \end{vmatrix}.$$

There is obviously no loss of generality in making B belong to the isobaric Tp series, since every other Tp series can be projected into it. The coordinates of the vertices of A and B are given by the determinants

$$\begin{matrix} A_1 \\ A_2 \\ A_3 \end{matrix} \begin{vmatrix} p_1a_1, & p_2a_2, & p_3a_3 \\ p_1a_2, & p_2a_3, & p_3a_1 \\ p_1a_3, & p_2a_1, & p_3a_2 \end{vmatrix} \equiv \Delta pa \quad \text{and} \quad \begin{matrix} B_1 \\ B_2 \\ B_3 \end{matrix} \begin{vmatrix} b_1, & b_2, & b_3 \\ b_2, & b_3, & b_1 \\ b_3, & b_1, & b_2 \end{vmatrix} \equiv \Delta b.$$

A and B are in direct triple perspective with C. If now it be supposed that A and B are also in direct triple perspective with each other, it follows from section II that the positive terms of the following determinant are equal,

$$\Delta pa \,.\, \Delta' b \equiv \begin{vmatrix} p_1a_1B_1+p_2a_2B_2+p_3a_3B_3, & p_1a_2B_1+p_2a_3B_2+p_3a_1B_3, & p_1a_3B_1+p_2a_1B_2+p_3a_2B_3 \\ p_1a_1B_2+p_2a_2B_3+p_3a_3B_1, & p_1a_2B_2+p_2a_3B_3+p_3a_1B_1, & p_1a_3B_2+p_2a_1B_3+p_3a_2B_1 \\ p_1a_1B_3+p_2a_2B_1+p_3a_3B_2, & p_1a_2B_3+p_2a_3B_1+p_3a_1B_2, & p_1a_3B_3+p_2a_1B_1+p_3a_2B_2 \end{vmatrix}$$

where the B's are the complementary minors of the b's in Δb, or, what is the same thing, the tangential coordinates of the sides of B. Hence if Qa and Qb be regarded as fixed, *i.e.*, if the a's and B's be regarded as constant, we have the following equations for the determination of $p_1 : p_2 : p_3$,

$$\begin{aligned} &(p_1a_1B_1+p_2a_2B_2+p_3a_3B_3)(p_1a_2B_2+p_2a_3B_3+p_3a_1B_1)(p_1a_3B_3+p_2a_1B_1+p_3a_2B_2) \\ &=(p_1a_1B_2+p_2a_2B_3+p_3a_3B_1)(p_1a_2B_3+p_2a_3B_1+p_3a_1B_2)(p_1a_3B_1+p_2a_1B_2+p_3a_2B_3) \\ &=(p_1a_1B_3+p_2a_2B_1+p_3a_3B_2)(p_1a_2B_1+p_2a_3B_2+p_3a_1B_3)(p_1a_3B_2+p_2a_1B_3+p_3a_2B_1)\,. \end{aligned}$$

That is, the point $P(p_1, p_2, p_3)$ may be any one of the nine points of intersection of the following system of cubics,

$$\begin{aligned} &(p_2^2p_3+p_3^2p_1+p_1^2p_2)(F_3-F_5)+(p_2p_3^2+p_3p_1^2+p_1p_2^2)(F_4-F_6) \\ &\qquad +p_1p_2p_3\{a_1^3(B_2^3-B_3^3)+a_2^3(B_3^3-B_1^3)+a_3^3(B_1^3-B_2^3)\}=0, \end{aligned}$$

$$\begin{aligned} &(p_2^2p_3+p_3^2p_1+p_1^2p_2)(F_5-F_1)+(p_2p_3^2+p_3p_1^2+p_1p_2^2)(F_6-F_2) \\ &\qquad +p_1p_2p_3\{a_1^3(B_3^3-B_1^3)+a_2^3(B_1^3-B_2^3)+a_3\,(B_2^3-B_3^3)\}=0, \end{aligned}$$

$$\begin{aligned} &(p_2^2p_3+p_3^2p_1+p_1^2p_2)(F_1-F_3)+(p_2p_3^2+p_3p_1^2+p_1p_2^2)(F_2-F_4) \\ &\qquad +p_1p_2p_3\{a_1^3(B_1^3-B_2^3)+a_2^3(B_2^3-B_3^3)+a_3^3(B_3^3-B_1^3)\}=0, \end{aligned}$$

where
$$\begin{aligned}
F_1 &\equiv a_2^2 a_3\, B_2^2 B_3 + a_3^2 a_1\, B_3^2 B_1 + a_1^2 a_2\, B_1^2 B_2\,,\\
F_2 &\equiv a_2\, a_3^2 B_2\, B_3^2 + a_3\, a_1^2 B_3\, B_1^2 + a_1\, a_2^2 B_1\, B_2^2,\\
F_3 &\equiv a_2^2 a_3\, B_3^2 B_1 + a_3^2 a_1\, B_1^2 B_2 + a_1^2 a_2\, B_2^2 B_3\,,\\
F_4 &\equiv a_2\, a_3^2 B_3\, B_1^2 + a_3\, a_1^2 B_1\, B_2^2 + a_1\, a_2^2 B_2\, B_3^2,\\
F_5 &\equiv a_2^2 a_3\, B_1^2 B_2 + a_3^2 a_1\, B_2^2 B_3 + a_1^2 a_2\, B_3^2 B_1\,,\\
F_6 &\equiv a_2\, a_3^2 B_1\, B_2^2 + a_3\, a_1^2 B_2\, B_3^2 + a_1\, a_2^2 B_3\, B_1^2.
\end{aligned}$$

It is at once evident that these three cubics are circumscribed to C. Hence P may coincide with any of the three vertices of C. Again, if $p_1 : p_2 : p_3 = \lambda_1 : \lambda_2 : \lambda_3$ be a solution of the cubics, it is evident that $p_1 : p_2 : p_3 = \lambda_2 : \lambda_3 : \lambda_1$ and $p_1 : p_2 : p_3 = \lambda_3 : \lambda_1 : \lambda_2$ are also solutions. We cannot have $\lambda_1 : \lambda_2 : \lambda_3 = \lambda_2 : \lambda_3 : \lambda_1 = \lambda_3 : \lambda_1 : \lambda_2$, except in the cases where $\lambda_1 : \lambda_2 : \lambda_3 = 1 : 1 : 1$ or $1 : \omega : \omega^2$ or $1 : \omega^2 : \omega$. But none of these is a solution of the cubics. Hence we conclude that of the six undetermined positions of P, if one be real all are real, and if one be imaginary all are imaginary, otherwise we should have the cubics intersecting in an odd number of imaginary points. Again, any two of the cubics are equivalent to two equations of the form

$$p_2^2 p_3 + p_3^2 p_1 + p_1^2 p_2 - k_1 p_1 p_2 p_3 = 0,$$

and $$p_2 p_3^2 + p_3 p_1^2 + p_1 p_2^2 - k_2 p_1 p_2 p_3 = 0,$$

where k_1, k_2 are constants which can readily be determined.

Eliminating p_1 from these equations, and removing the factors which indicate that the cubics are circumscribed to C, we obtain the sextic

$$(p_2^3 - k_2 p_2^2 p_3 + k_1 p_2 p_3^2 - p_3^3)^2 = 0.$$

The fact that the expression on the left hand is a complete square indicates that the three cubics have contact of the first kind with each other at three points, *i.e.*, that the six undetermined positions of P reduce to three distinct points. Again, since the equation

$$p_2^3 - k_2 p_2^2 p_3 + k_1 p_2 p_3^2 - p_3^3 = 0$$

is known to have at least one real solution, giving a real position of P, it follows from what has been already said that the other positions of P are also real. Hence we conclude that the three cubics intersect in general in six real and distinct points, three of which, however, coincide with the vertices of C. When P coincides with a vertex of C, the triangle A obviously reduces to a point-triangle coinciding with the same vertex. Hence, excluding point-

triangles, we may say that in general and in the absence of any particular relations connecting the a's of A with the b's of B, there are three and only three positions of P which give us a triangle A directly triply in perspective with B, and satisfying the condition that it shall belong to a certain Tq series different from that of B, and at the same time to some Tp series (distinct for each of the three cases) different from that of B. Thus on the diagram there are on the horizontal line marked Qa, three positions of A, none of them on the same vertical line with B, which represent triangles directly in triple perspective with the triangle represented by B.

I now turn to the special case where A and B, while still belonging to different Tp series, belong to the same Tq series.

If we retain the same coordinates as before for the vertices of A, those of B, which may now be denoted by A′ (see diagram) become

$$\begin{array}{c} A_1' \\ A_2' \\ A_3' \end{array} \begin{vmatrix} a_1, & a_2, & a_3 \\ a_2, & a_3, & a_1 \\ a_3, & a_1, & a_2 \end{vmatrix} \equiv \Delta a.$$

We therefore obtain for the determination of the values $p_1 : p_2 : p_3$ the same system of intersecting cubics as we obtained on page 125, except that in the values there given for the F's, and in the coefficients of $p_1 p_2 p_3$, A is everywhere substituted for B, A_κ being the complementary minor of a_κ in Δa.

The resulting relations* which can be established between the

---

* For example, distinguishing the new F's by dashes, and remembering that

$$\left.\begin{aligned} A_\kappa &\equiv a_\lambda a_\mu - a^2_\kappa, \\ \text{and} \quad A_\lambda A_\mu - A^2_\kappa &\equiv a_\kappa \, . \, \Delta a, \end{aligned}\right. \quad \begin{pmatrix} \kappa = 1, 2, 3 \\ \lambda = 2, 3, 1 \\ \mu = 3, 1, 2 \end{pmatrix}$$

we have

$$F_5' - F_1' \equiv -(F_2' - F_4') \equiv -a_1 a_2 a_3 \, . \, \Delta a^2,$$

$$F_1' - F_3' \equiv -(F_6' - F_2') \equiv -(a_1 a_2 a_3 \, . \, \Sigma a_1^3 - \Sigma a_2^3 a_3^3) \Delta a,$$

$$F_3' - F_5' \equiv -(F_4' - F_6') \equiv (F_6' - F_2') - (F_5' - F_1') \equiv (F_2' - F_4') - (F_1' - F_3')$$

$$\equiv a_1^3 a_2^3 a_3^3 \begin{vmatrix} 1/a_1, & 1/a_2, & 1/a_3 \\ 1/a_2, & 1/a_3, & 1/a_1 \\ 1/a_3, & 1/a_1, & 1/a_2 \end{vmatrix} . \, \Delta a,$$

$$\Sigma a_1^3 (A_3^3 - A_1^3) \equiv -\Sigma a_1^3 (A_1^3 - A_2^3) \equiv (\Sigma A_1^6 - \Sigma A_2^3 A_3^3) / \Delta a,$$

$$\text{and} \quad \Sigma a_1^3 (A_2^3 - A_3^3) \equiv 0.$$

coefficients of the cubics, though not devoid of interest in themselves, do not seem to affect the result already obtained in the more general case, viz., that the three cubics intersect in three real and distinct points, in addition to the vertices of C. Hence we conclude that every triangle of a Tq series is directly triply in perspective with three other triangles, point-triangles being excluded, of the same series.

I shall now discuss the second case of the general problem, where A and B, two triangles belonging to different Tp and Tq series, are supposed to be in triple perspective with each other not directly but perversely. In this case, the coordinates of the vertices of A and B being the same as before (see page 124), we have the negative terms of the determinant $\Delta pa . \Delta' b$ equal. Hence we have the following equations for the determination of $p_1 : p_2 : p_3$, the a's and the B's being regarded as constant,

$$\begin{aligned}
&(p_1 a_1 \mathrm{B}_1 + p_2 a_2 \mathrm{B}_2 + p_3 a_3 \mathrm{B}_3)(p_1 a_2 \mathrm{B}_3 + p_2 a_3 \mathrm{B}_1 + p_3 a_1 \mathrm{B}_2)(p_1 a_3 \mathrm{B}_2 + p_2 a_1 \mathrm{B}_3 + p_3 a_2 \mathrm{B}_1) \\
= &(p_1 a_1 \mathrm{B}_2 + p_2 a_2 \mathrm{B}_3 + p_3 a_3 \mathrm{B}_1)(p_1 a_2 \mathrm{B}_1 + p_2 a_3 \mathrm{B}_2 + p_3 a_1 \mathrm{B}_3)(p_1 a_3 \mathrm{B}_3 + p_2 a_1 \mathrm{B}_1 + p_3 a_2 \mathrm{B}_2) \\
= &(p_1 a_1 \mathrm{B}_3 + p_2 a_2 \mathrm{B}_1 + p_3 a_3 \mathrm{B}_2)(p_1 a_2 \mathrm{B}_2 + p_2 a_3 \mathrm{B}_3 + p_3 a_1 \mathrm{B}_1)(p_1 a_3 \mathrm{B}_1 + p_2 a_1 \mathrm{B}_2 + p_3 a_2 \mathrm{B}_3).
\end{aligned}$$

That is, the point P(p_1, p_2, p_3) may be any one of the nine points of intersection of the following system of cubics,

$$\begin{aligned}
&p_2^2 p_3 (f_3 - f_5) + p_3^2 p_1 (f_5 - f_1) + p_1^2 p_2 (f_1 - f_3) \\
&+ p_2 p_3^2 (f_4 - f_6) + p_3 p_1^2 (f_6 - f_2) + p_1 p_2^2 (f_2 - f_4) = 0,
\end{aligned}$$

$$\begin{aligned}
&p_2^2 p_3 (f_5 - f_1) + p_3^2 p_1 (f_1 - f_3) + p_1^2 p_2 (f_3 - f_5) \\
&+ p_2 p_3^2 (f_6 - f_2) + p_3 p_1^2 (f_2 - f_4) + p_1 p_2^2 (f_4 - f_6) = 0,
\end{aligned}$$

$$\begin{aligned}
&p_2^2 p_3 (f_1 - f_3) + p_3^2 p_1 (f_3 - f_5) + p_1^2 p_2 (f_5 - f_1) \\
&+ p_2 p_3^2 (f_2 - f_4) + p_3 p_1^2 (f_4 - f_6) + p_1 p_2^2 (f_6 - f_2) = 0,
\end{aligned}$$

where $f_1 \equiv a_2^2 a_3 \mathrm{B}_2^2 \mathrm{B}_3 + a_3^2 a_1 \mathrm{B}_1^2 \mathrm{B}_2 + a_1^2 a_2 \mathrm{B}_3^2 \mathrm{B}_1$,

$f_2 \equiv a_2 a_3^2 \mathrm{B}_1 \mathrm{B}_2^2 + a_3 a_1^2 \mathrm{B}_3 \mathrm{B}_1^2 + a_1 a_2^2 \mathrm{B}_2 \mathrm{B}_3^2$,

$f_3 \equiv a_2^2 a_3 \mathrm{B}_1^2 \mathrm{B}_2 + a_3^2 a_1 \mathrm{B}_3^2 \mathrm{B}_1 + a_1^2 a_2 \mathrm{B}_2^2 \mathrm{B}_3$,

$f_4 \equiv a_2 a_3^2 \mathrm{B}_3 \mathrm{B}_1^2 + a_3 a_1^2 \mathrm{B}_2 \mathrm{B}_3^2 + a_1 a_2^2 \mathrm{B}_1 \mathrm{B}_2^2$,

$f_5 \equiv a_2^2 a_3 \mathrm{B}_3^2 \mathrm{B}_1 + a_3^2 a_1 \mathrm{B}_2^2 \mathrm{B}_3 + a_1^2 a_2 \mathrm{B}_1^2 \mathrm{B}_2$,

and $f_6 \equiv a_2 a_3^2 \mathrm{B}_2 \mathrm{B}_3^2 + a_3 a_1^2 \mathrm{B}_1 \mathrm{B}_2^2 + a_1 a_2^2 \mathrm{B}_3 \mathrm{B}_1^2$.

Six of the points of intersection of the above cubics are evident from inspection, viz., the three vertices of the fundamental triangle C, and the points $(1, 1, 1)$, $(1, \omega, \omega^2)$, and $(1, \omega^2, \omega)$. When P coincides with any one of the vertices of C, the triangle A, as we have already seen, reduces to that vertex. When it coincides with the point $(1, 1, 1)$, A belongs to the same Tp series as B, which is contrary to the supposition. When it coincides with either of the points $(1, \omega, \omega^2)$, $(1, \omega^2, \omega)$, A again belongs, as can readily be shown, to the same Tp series as B, and, further, has imaginary vertices. Thus the six positions of P which have been found give us no real triangle A, with distinct vertices and belonging to a different Tp series from B.

With regard to the remaining three positions of P, it is evident from the equations that if $p_1 : p_2 : p_3 = \lambda_1 : \lambda_2 : \lambda_3$ be one solution, then $p_1 : p_2 : p_3 = \lambda_2 : \lambda_3 : \lambda_1$ and $p_1 : p_2 : p_3 = \lambda_3 : \lambda_1 : \lambda_2$ are also solutions. Hence these three positions of P must all be real, for if one were imaginary the remaining two would be so also, and the cubics would intersect in four real and five imaginary points. Again, the three undetermined positions of P do not in general coincide with any one of the six positions that have been determined, for it can be shown that the cubics do not touch each other at any of these six points, and that, in the absence of particular relations connecting the a's and the B's, no one of these six points is a double point. Further, the three undetermined positions do not in general coincide with each other, for in order that $\lambda_1 : \lambda_2 : \lambda_3 = \lambda_2 : \lambda_3 : \lambda_1 = \lambda_3 : \lambda_1 : \lambda_2$, we must have $\lambda_1 : \lambda_2 : \lambda_3 = 1 : 1 : 1$ or $1 : \omega : \omega^2$ or $1 : \omega^2 : \omega$. That is, the three undetermined positions cannot coincide with each other unless they coincide with one of the points $(1, 1, 1)$, $(1, \omega, \omega^2)$, $(1, \omega^2, \omega)$, which, as we have just seen, is not the case.

Thus we are led to the conclusion that in general there are three and only three positions of P which give us a real triangle A in perverse triple perspective with B, possessing distinct vertices and satisfying the condition that it shall belong to a certain Tq series different from that of B and at the same time to some Tp series (distinct for each of the three cases) different from that of B. Combining this with the result obtained in the case when A and B were supposed to be in direct triple perspective, we conclude that if any two Tq series be selected, every triangle of the one is in

triple perspective with six triangles of the other, the restriction that triangles belonging to the same Tp series be excluded from consideration being observed. When this restriction is removed we can say that every triangle of a Tq series is in triple perspective with seven triangles of any other Tq series, directly with three and perversely with four.

In the special case when A and B, while still belonging to different Tp series, belong to the same Tq series, we may as before suppose that B becomes A′, the coordinates of whose vertices are given by the rows of Δa. In this case the three cubics which by their intersection determine the nine possible positions of P are the same as those already given on page 128, except that A must be substituted throughout for B in the values there given for the f's. If the new f's be distinguished by dashes, we readily verify that

$$f_1' - f_3' \equiv -a_1(a_2^3 - a_3^3)A_1 \,.\, \Delta a \equiv f_4' - f_6',$$

$$f_3' - f_5' \equiv -a_2(a_3^3 - a_1^3)A_2 \,.\, \Delta a \equiv f_6' - f_2',$$

$$\text{and} \quad f_5' - f_1' \equiv -a_3(a_1^3 - a_2^3)A_3 \,.\, \Delta a \equiv f_2' - f_4'.$$

These relations between the elements and the complementary minors of a persymmetric determinant of the type Δa seem in themselves noteworthy. They lead at once to the further identity

$$a_1(a_2^3 - a_3^3)A_1 + a_2(a_3^3 - a_1^3)A_2 + a_3(a_1^3 - a_2^3)A_3 \equiv 0.$$

The identities obviously still hold when, by having two of its rows interchanged, Δa is converted into a circulant of the third order.

As a result of these identities the equations of the three intersecting cubics may be written

$$U_1 \equiv p_1(p_2^2 + p_3^2)a_3 + p_2(p_3^2 + p_1^2)a_1 + p_3(p_1^2 + p_2^2)a_2 = 0,$$

$$U_2 \equiv p_1(p_2^2 + p_3^2)a_1 + p_2(p_3^2 + p_1^2)a_2 + p_3(p_1^2 + p_2^2)a_3 = 0,$$

$$\text{and} \quad U_3 \equiv p_1(p_2^2 + p_3^2)a_2 + p_2(p_3^2 + p_1^2)a_3 + p_3(p_1^2 + p_2^2)a_1 = 0,$$

where $a_1 + a_2 + a_3 \equiv 0$.

U_1, U_2, U_3 evidently intersect in the same six points as the cubics obtained in connection with the more general case, viz., in the three vertices of C, and in the points $(1, 1, 1)$, $(1, \omega, \omega^2)$, $(1, \omega^2, \omega)$.

Further,

$$\frac{dU_\kappa}{dp_1} \equiv (p_2^2+p_3^2)a_\mu + 2p_1p_2a_\kappa + 2p_3p_1a_\lambda,$$

$$\frac{dU_\kappa}{dp_2} \equiv 2p_1p_2a_\mu + (p_3^2+p_1^2)a_\kappa + 2p_2p_3a_\lambda,$$

and $$\frac{dU_\kappa}{dp_3} \equiv 2p_3p_1a_\mu + 2p_2p_3a_\kappa + (p_1^2+p_2^2)a_\lambda,$$

$(\kappa = 1, 2, 3;\ \lambda = 2, 3, 1;\ \mu = 3, 1, 2)$.

Therefore when $p_1 = p_2 = p_3 = 1$,

$$\frac{dU_\kappa}{dp_1} = \frac{dU_\kappa}{dp_2} = \frac{dU_\kappa}{dp_3} = 2(a_\kappa + a_\lambda + a_\mu) = 0.$$

U_1, U_2, U_3, consequently, have a double point at (1, 1, 1). Their nine points of intersection have therefore been determined, since the double point counts as four intersections. It thus appears that if we exclude imaginary and point triangles, we have only one triangle A (viz., that derived from A′ by the multiplication of the coordinates of its vertices by (1, 1, 1)) which belongs to the same Tq series as A′ and is yet in perverse triple perspective with it. But since in this case A is just the same as A′, it also must be excluded. Thus we reach the conclusion that in any Tq series (that series alone excepted whose constituent triangles all coincide with the fundamental triangle C) no two triangles are in perverse triple perspective with each other. Combining this with the result obtained for the case when A and A′ were supposed to be in direct triple perspective, we conclude that in any Tq series every triangle is in triple perspective with three and no more than three other triangles of the same series, and that the perspective is direct.

It has been proved that in general every triangle of a Tq series is in triple perspective with seven and no more than seven triangles of any other Tq series. When certain particular relations exist between the coordinates of the two triangles which define the two Tq series, this number may be exceeded. A case of this arises in the following way.

Let A′ be the isobaric triangle and A_1 any other triangle of the Tq series defined by the triangle whose vertices have for coordinates

the rows of the determinant Δa, *i.e.*, by A'. This series I shall call the Tq (a) series. The coordinates of the vertices of A' and A_1 are

$$\begin{array}{c|ccc|}
A_1' & a_1, & a_2, & a_3 \\
A_2' & a_2, & a_3, & a_1 \\
A_3' & a_3, & a_1, & a_2
\end{array}, \quad
\begin{array}{c|ccc|}
A_{11} & p_1a_1, & p_2a_2, & p_3a_3 \\
A_{12} & p_1a_2, & p_2a_3, & p_3a_1 \\
A_{13} & p_1a_3, & p_2a_1, & p_3a_2
\end{array}.$$

Let B_1 be the triangle obtained by joining the corresponding vertices of A' and A. The equations of the sides of B_1 will be

$$A_1'A_{11} \equiv x_1(p_2 - p_3)/a_1 + x_2(p_3 - p_1)/a_2 + x_3(p_1 - p_2)/a_3 = 0,$$
$$A_2'A_{12} \equiv x_1(p_2 - p_3)/a_2 + x_2(p_3 - p_1)/a_3 + x_3(p_1 - p_2)/a_1 = 0,$$
$$A_3'A_{13} \equiv x_1(p_2 - p_3)/a_3 + x_2(p_3 - p_1)/a_1 + x_3(p_1 - p_2)/a_2 = 0.$$

The form of the coefficients shows, as Mr Ferrari has pointed out, that B_1 is in direct triple perspective with the fundamental triangle C. By forming the coordinates of its vertices we find that it belongs to the Tq series defined by the triangle

$$\begin{vmatrix}
\dfrac{A_1}{a_1}, & \dfrac{A_2}{a_2}, & \dfrac{A_3}{a_3} \\
\dfrac{A_2}{a_2}, & \dfrac{A_3}{a_3}, & \dfrac{A_1}{a_1} \\
\dfrac{A_3}{a_3}, & \dfrac{A_1}{a_1}, & \dfrac{A_2}{a_2}
\end{vmatrix},$$

a series which I shall denote by the symbol T$q(b)$, and to the Tp series defined by the point

$$P\left(\frac{1}{p_2 - p_3}, \ \frac{1}{p_3 - p_1}, \ \frac{1}{p_1 - p_2}\right).$$

An infinity of values of $\lambda_1 : \lambda_2 : \lambda_3$ can be found to satisfy the equation

$$\lambda_1(p_2 - p_3) + \lambda_2(p_3 - p_1) + \lambda_3(p_1 - p_2) = 0.$$

For all these values,

$\lambda_1 a_1$, $\lambda_2 a_2$, $\lambda_3 a_3$ satisfy the equation of the first side of B_1,

$\lambda_1 a_2$, $\lambda_2 a_3$, $\lambda_3 a_1$ satisfy the equation of the second side of B_1,

and $\lambda_1 a_3$, $\lambda_2 a_1$, $\lambda_3 a_2$ satisfy the equation of the third side of B_1.

That is to say, B_1 is circumscribed not only to the triangles A' and A_1 of the T$q(a)$ series, but to a whole infinity of triangles of that series.

Again, by joining the vertices of A' to the corresponding vertices of A_2 any other triangle of the $Tq\ (a)$ series which is not inscribed in B_1, we obtain another triangle B_2 which also belongs to the $Tq(b)$ series, and is circumscribed to an infinity of triangles of the $Tq(a)$ series. Proceeding in this way we can form an infinity of triangles, $B_1,\ B_2,\ B_3 \ldots B_\infty$, all belonging to the series $Tq(b)$ and each of them circumscribed to an infinity of triangles of the series $Tq(a)$, thus exhausting all the triangles of that series. The triangles $B_1,\ B_2,\ B_3 \ldots B_\infty$ are all circumscribed to the triangle A'. By starting with some other triangle of the $Tq(a)$ series than A', and by joining its vertices to the corresponding vertices of other triangles of the series, we find in the same way that it also is inscribed in an infinity of triangles belonging to the $Tq(b)$ series. Hence we reach the conclusion that every triangle of the $Tq(b)$ series is circumscribed to an infinity of triangles of the $Tq(a)$ series, and that every triangle of the latter series is inscribed in an infinity of triangles of the former. Since two triangles such that one is inscribed in the other are necessarily in triple perspective with each other, it follows that for every two Tq series, the coordinates of the vertices of whose defining triangles are related as in $Tq(a)$ and $Tq(b)$, every triangle of the one is in triple perspective (perversely) with an infinity of triangles of the other.

The coordinates given above for P in the case of the triangle B_1 show that for the triangles of the subseries $B_1,\ B_2,\ B_3 \ldots B_\infty$, the locus of P is the circumconic

$$\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}=0.$$

Three of the triangles of the series $Tq(b)$ reduce to point-triangles coinciding with the vertices of the fundamental triangle C, viz., the three triangles obtained by making P coincide with these vertices in succession. It is easy to see that these are the only point-triangles in the series $Tq(b)$. Since A' is in direct triple perspective, and consequently in perspective of the first kind, with three other triangles of the series $Tq(a)$, three of the triangles of the subseries $B_1,\ B_2,\ B_3 \ldots B_\infty$ must reduce to points, viz., the centres of perspective corresponding to these three perspectives of the first kind. Hence the three point-triangles of the series $Tq(b)$ are included in the subseries $B_1,\ B_2,\ B_3 \ldots B_\infty$, and consequently in every other such

subseries of $Tq(b)$. Therefore we infer that every triangle of the $Tq(a)$ series, and consequently of every other Tq series, is in simple perspective of the first kind with three infinities of triangles of the same series, with respect to the three vertices of C respectively as common centres of perspective.

A fuller exploration of the ∞^4 triangles in triple perspective with a given triangle would, no doubt, lead to many other interesting relations subsisting between them, but for the present those given may suffice.

A considerable number of elementary theorems on determinants arise in connection with the study of these triangles. The following contingent identities connecting the elements a_κ and the complementary minors A_κ of a persymmetric determinant of three elements, a_1, a_2, a_3, and consequently of the circulant obtained by transposing two of its rows, may serve as a specimen. Let A and A′ be two triangles belonging to the same Tq series, the latter being the isobaric or defining triangle of the series, and let the coordinates of their vertices be given by the rows of the following determinants,

$$\mathrm{A}\ \begin{vmatrix} p_1a_1, & p_2a_2, & p_3a_3 \\ p_1a_2, & p_2a_3, & p_3a_1 \\ p_1a_3, & p_2a_1, & p_3a_2 \end{vmatrix} \equiv \Delta pa, \qquad \mathrm{A}'\ \begin{vmatrix} a_1, & a_2, & a_3 \\ a_2, & a_3, & a_1 \\ a_3, & a_1, & a_2 \end{vmatrix} \equiv \Delta a.$$

Then the determinant $\Delta pa\,.\,\Delta' a$, obtained by multiplying by rows Δpa and the adjugate of Δa, has for its first, second, and third positive terms, and its first, second, and third negative terms respectively

$$\left.\begin{aligned} \alpha_1 &\equiv \Pi(p_1a_\kappa A_\kappa + p_2a_\lambda A_\lambda + p_3a_\mu A_\mu), \\ \alpha_2 &\equiv \Pi(p_1a_\kappa A_\lambda + p_2a_\lambda A_\mu + p_3a_\mu A_\kappa), \\ \alpha_3 &\equiv \Pi(p_1a_\kappa A_\mu + p_2a_\lambda A_\kappa + p_3a_\mu A_\lambda), \end{aligned}\right\} \left(\begin{aligned} \kappa &= 1, 2, 3 \\ \lambda &= 2, 3, 1 \\ \mu &= 3, 1, 2 \end{aligned}\right)$$

$$\alpha_4 \equiv (p_1a_1A_1 + p_2a_2A_2 + p_3a_3A_3)(p_1a_2A_3 + p_2a_3A_1 + p_3a_1A_2)(p_1a_3A_2 + p_2a_1A_3 + p_3a_2A_1),$$

$$\alpha_5 \equiv (p_1a_1A_2 + p_2a_2A_3 + p_3a_3A_1)(p_1a_2A_1 + p_2a_3A_2 + p_3a_1A_3)(p_1a_3A_3 + p_2a_1A_1 + p_3a_2A_2),$$

$$\alpha_6 \equiv (p_1a_1A_3 + p_2a_2A_1 + p_3a_3A_2)(p_1a_2A_2 + p_2a_3A_3 + p_3a_1A_1)(p_1a_3A_1 + p_2a_1A_2 + p_3a_2A_3)$$

The tangential coordinates of the sides of A are given by the rows of the determinant

$$\begin{vmatrix} \dfrac{A_1}{p_1}, & \dfrac{A_2}{p_2}, & \dfrac{A_3}{p_3} \\ \dfrac{A_2}{p_1}, & \dfrac{A_3}{p_2}, & \dfrac{A_1}{p_3} \\ \dfrac{A_3}{p_1}, & \dfrac{A_1}{p_2}, & \dfrac{A_2}{p_3} \end{vmatrix},$$

which though not strictly speaking the adjugate of Δpa, being in fact the adjugate after each row has been divided by $p_1 p_2 p_3$, may be denoted by $\Delta' pa$. The first, second, and third positive terms and the first, second, and third negative terms of the determinant $\Delta a \,.\, \Delta' pa$, where the multiplication is effected in precisely the same manner as in the case of $\Delta pa \,.\, \Delta' a$, are easily seen to be exactly the same as α_1, α_2, α_3, α_4, α_5, α_6, respectively, except that p_κ is everywhere replaced by $1/p_\kappa$. They may be denoted by α_1', α_2', α_3', α_4', α_5', α_6'. Now if A and A′ be in perspective in any way, two of the α's must be equal, and also two of the α''s. For example, suppose that we have the perspective

$$\frac{A_1\,A_2\,A_3}{A_2'A_3'A_1'}.\quad \text{This is the same as the perspective}\quad \frac{A_1'A_2'A_3'}{A_3\,A_1\,A_2}.$$

Hence in this case $\alpha_3 = \alpha_1$, and also $\alpha_1' = \alpha_2'$. Hence, generally, if $\alpha_1 = \alpha_2$, then $\alpha_3' = \alpha_1'$, if $\alpha_2 = \alpha_3$, $\alpha_3' = \alpha_2'$, if $\alpha_3 = \alpha_1$, then $\alpha_1' = \alpha_2'$, if $\alpha_4 = \alpha_5$, then $\alpha_6' = \alpha_4'$, if $\alpha_5 = \alpha_6$, then $\alpha_6' = \alpha_5'$, if $\alpha_6 = \alpha_4$, then $\alpha_4' = \alpha_5'$, and conversely. We have already seen that the relation $\alpha_4 = \alpha_5 = \alpha_6$, and consequently also the relation $\alpha_4' = \alpha_5' = \alpha_6'$, can be true, zero values of the p's being excluded, only when

$$p_1 : p_2 : p_3 = 1 : 1 : 1 \text{ or } 1 : \omega : \omega^2 \text{ or } 1 : \omega^2 : \omega.$$

It may be worth stating as affording an exercise in determinants that if we eliminate p_1 by the dialytic method from the cubic equations U_1, U_2 given on page 130, and remove the factor $p_2 p_3$, we obtain the determinant

$$\begin{vmatrix} 0 & , \; p_2\alpha_1 + p_3\alpha_2 \;, & (p_2^2 + p_3^2)\alpha_3 & , \; p_2\alpha_2 + p_3\alpha_1 \\ p_3\alpha_1 + p_3\alpha_2, & (p_2^2 + p_3^2)\alpha_3, & p_2 p_3(p_2\alpha_2 + p_3\alpha_1), & 0 \\ p_2\alpha_2 + p_3\alpha_3, & (p_2^2 + p_3^2)\alpha_1, & p_2 p_3(p_2\alpha_3 + p_3\alpha_2), & 0 \\ 0 & , \; p_2\alpha_2 + p_3\alpha_3 \;, & (p_2^2 + p_3^2)\alpha_1 & , \; p_2\alpha_3 + p_3\alpha_2 \end{vmatrix},$$

which, since the cubics have a double point at (1, 1, 1), and intersect besides at (1, ω, ω^2) and (1, ω^2, ω), must be equal to $(p_2 - p_3)^3(p_2^3 - p_3^3)$, when the condition

$$a_1 + a_2 + a_3 = 0,$$

is fulfilled.

IV.

I conclude with a note on a paper by Mr Eugen Jahnke in *Crelle's Journal* (Bd. 123, Heft 1, p. 42) entitled "Eine dreifach perspectiven Dreiecken zugehörige Punktgruppe," in which the author studies a group of 9 points, V_{ik}, called by him the *Veronese* points, arising in connection with two directly perspective triangles $A_1A_2A_3$ and $B_1B_2B_3$, and defined as follows, the brackets containing the pairs of lines whose intersections give the points:—

$$V_{11}(A_2B_3,\ A_3B_2),\quad V_{12}(A_3B_2,\ A_1B_1),\quad V_{13}(A_1B_1,\ A_2B_3),$$
$$V_{21}(A_2B_2,\ A_3B_1),\quad V_{22}(A_3B_1,\ A_1B_3),\quad V_{23}(A_1B_3,\ A_2B_2),$$
$$V_{31}(A_2B_1,\ A_3B_3),\quad V_{32}(A_3B_3,\ A_1B_2),\quad V_{33}(A_1B_2,\ A_2B_1).$$

If C_1, C_2, C_3 be the centres of perspective of $A_1A_2A_3$ and $B_1B_2B_3$ (C_1 lying on A_1B_1, C_2 on A_2B_2, and C_3 on A_3B_3), then, as is well known, the three triangles A, B, C are triply in perspective in pairs, the centres of perspective of each two being the vertices of the third. Triangles related in this way are said by Mr Jahnke to be in *desmic* position.

A further group of 18 points, P_i, Q_i, R_i, X_i, Y_i, Z_i ($i = 1, 2, 3$) arises in connection with the following theorem given by Mr Jahnke:

$A_1A_2A_3$ is in desmic position with the six pairs of triangles

$$V_{11}V_{22}V_{33},\ P_1C_3C_2;\quad V_{12}V_{23}V_{31},\ C_1C_3P_2;\quad V_{13}V_{21}V_{32},\ C_1P_3C_2;$$
$$V_{11}V_{32}V_{23},\ X_1B_2B_3;\quad V_{13}V_{31}V_{22},\ B_1X_2B_3;\quad V_{12}V_{33}V_{21},\ B_1B_2X_3;$$

$B_1B_2B_3$ is in desmic position with the six pairs of triangles

$$V_{11}V_{22}V_{33},\ Q_1C_2C_3;\quad V_{12}V_{23}V_{31},\ C_1Q_2C_3;\quad V_{13}V_{21}V_{32},\ C_1C_2Q_3;$$
$$V_{11}V_{31}V_{21},\ Y_1A_2A_3;\quad V_{12}V_{32}V_{22},\ A_1Y_2A_3;\quad V_{13}V_{33}V_{23},\ A_1A_2Y_3;$$

and $C_1C_2C_3$ is in desmic position with the six pairs of triangles

$$V_{11}V_{31}V_{21},\ R_1A_3A_2;\quad V_{12}V_{32}V_{22},\ A_1A_3R_2;\quad V_{13}V_{23}V_{33},\ A_1R_3A_2;$$
$$V_{11}V_{23}V_{32},\ Z_1B_2B_3;\quad V_{13}V_{22}V_{31},\ B_1Z_2B_3;\quad V_{12}V_{21}V_{33},\ B_1B_2Z_3.$$

Mr Jahnke gives numerous theorems regarding the V-points, and one regarding the points of the second group, viz., that the following triads of points are collinear:

$$C_i,\ P_i,\ Q_i;\quad A_i,\ Y_i,\ R_i;\quad B_i,\ Z_i,\ X_i\qquad (i=1,\ 2,\ 3).$$

I add the following:

(1) The triangles A, B, C, P, Q, R, X, Y, Z are triply in perspective with each other in pairs, and in fact belong to the same Tp series with respect to any one of them as triangle of reference;

(2) The triads A, P, X; B, Q, Y: C, R, Z are in desmic position;

(3) The three triangles $V_{11}V_{22}V_{33}$, $V_{23}V_{31}V_{12}$, $V_{32}V_{13}V_{21}$ belong to the same Tq series with respect to A. Since, as Mr Jahnke has shown, these triangles are also in direct triple perspective with B, we have here an illustration of the fact already proved, that a triangle of one Tq series is in direct triple perspective with three triangles of any other Tq series.

The proof of these propositions, which I omit, may be obtained by projecting the figure so that the coordinates of the vertices of B with respect to A are given by the rows of a determinant of the form

$$\begin{vmatrix} b_1, & b_2, & b_3 \\ b_2, & b_3, & b_1 \\ b_3, & b_1, & b_2 \end{vmatrix},$$

and by calculating the coordinates of the vertices of the other triangles involved. It will be found that the coordinates of the vertices of P, Q, R, X, Y, Z are the rows of determinants which either are circulants or can be converted into circulants by interchanging the last two rows.

A Mechanical Construction for the Quartic Trisectrix.

By H. Poole.

Figure 46.

The model consists of a circular template, of radius a, hinged at O, a point on its circumference, to a bar OA, where $OA = a$. P is the centre of the circle, and E is the middle point of OP. C is a point on the circumference such that $OP = OC = a$, so that CE is perpendicular to OP.

From E two arms EX and EY radiate, and are so arranged by a linkage that the angles YEC and XEC are equal.

To trisect an angle with the instrument, place OA along one side, make the arm EX fall on A, and open or close AOC, EX running along A, until the point where the arm EY cuts the circumference of the template lies on the other side of the angle. Let this point be called B.

Then OC is one of the trisectors of the angle AOB.

Since $OP = OC = OA = a$,

$\therefore$ A, C, P are on the circumference of a circle equal to OCB, which cuts it at C, and has O for its centre.

Now CE is perpendicular to OP, which joins the centres.

$\therefore$ arcCB = arcCA.

$\therefore$ angles CPB and COA are equal.

But angle CPB = 2 . angle COB,

$\therefore$ angle COA = 2 . angle COB,

or OC is one of the trisectors of the angle AOB.

Figure 47.

To draw the curve, let OA be fixed, and let the arm EX always run along A: then the point B describes the Quartic Trisectrix.

Let B be any point on the curve.

With centre O, and radius OA describe a circle ACP.

Join BA, cutting the circle ACP at Q : join OQ.
Then, obviously, OQ = OA, and the angles OQA and OAQ are equal. Also, from the congruence of the triangles BPE and AOE, we see that PBAO is a trapezium.

Hence the angles OQA, OAQ, and PBQ are equal.

Hence OQ is equal and parallel to PB.
Therefore QBPO is a parallelogram, and QB = OP and is constant.

It follows that the locus of B is that particular kind of Limaçon which is called the Quartic Trisectrix.

FIGURE 48.

Another mechanical method of drawing the Limaçon is given by the linkage shown in the figure.

Here $AB = OC = a$, $OA = OB = DE = 2a$, $CD = BE = 4a$, and, in the case of the trisectrix, $DP = OD$. O and A are fixed, and P describes the curve.

From the arrangement it will be seen that $\angle AOC = \angle CDE$.

$$\therefore \quad \angle AOD = \angle ODP.$$

Thus P moves so that $OD = DP$ and $\angle AOD = \angle ODP$, thus describing the trisectrix.

Other Limaçons may also be drawn, the eccentricity depending on the length of DP.

Note on Mental Division by Large Numbers.

By J. Taylor, M.A.

Since $\frac{A}{B} = \frac{nA}{nB}$, it is possible to divide mentally by many numbers, integral or fractional.

Examples: $\frac{3275}{125} = \frac{8.3275}{8.\ 125} = \frac{26200}{1000} = \mathbf{26 \cdot 2}$;

$$\frac{4579}{14\frac{2}{7}} = \frac{7.4579}{7.\ 14\frac{2}{7}} = \frac{32053}{100} = \mathbf{320 \cdot 53}.$$

In 1901, when I was drawing up notes on Mental Arithmetic, I looked into many text-books in search of a simple method for dividing by such numbers as 19, 29, 99, 87, etc., but found none. The following method, viz., *that of using a multiple of ten as divisor instead of a given divisor*, was then discovered by me, and I think it simple enough to be learned and practised by any one.

If it be required to divide A by D, let the quotient at any step be Q, then the product at that step will be DQ, and the remainder, $R = A - DQ$, where R cannot exceed D nor be less than zero.

Instead of working with D as divisor, take as divisor d, *that multiple of ten which is nearest to D*, and if, at each step, care be taken with Q, so that R never exceeds D nor is less than zero, the required quotient Q will be obtained.

Dividing by d, the remainder at any step is $r = A - dQ$,

and since $\quad R = A - DQ$, and $r = A - dQ$,

$$\therefore \quad R = r + (d - D)Q.$$

Thus R, the remainder which would have been obtained on dividing by D, is obtained at every step by adding $(d - D)Q$ to r, the remainder obtained on dividing by d. The importance of obtaining R correctly at each step is so great that I would suggest that the method of obtaining it in each sum be made the *key-word of that sum.*

For example: In dividing by 19, 29, 39, 49, or 99, the divisor used is 20, 30, 40, 50, or 100, where $d - D = 1$, so that $R = r + Q$, and the key-word in such examples would be: *Add once Q.*

In dividing by 67, 87 or 97, the divisor used is 70, 90 or 100, where $d - D = 3$, and $R = r + 3Q$, and the key-word would be: *Add three times Q.*

Again, in dividing by 31, 61 or 71 the divisor used would be 30, 60 or 70, where $d - D = -1$, and $R = r - Q$, and the key-word would be: *Subtract once Q.*

In 43, 53, 73 or 83 the key-word would be: *Subtract three times Q.*

The divisors used in working the following examples are printed in heavier type.

20 19 | 451360 Key-word: *Add once Q.*

$23755\frac{15}{19}$

Divide by 20	Quot.	$r + Q$	$= R$
45	2	5 + 2	7
71	3	11 + 3	14
143	7	3 + 7	10
106	5	6 + 5	11
110	5	10 + 5	15

100 99 | 223134 — **$2253\frac{87}{99}$**

80 79 | 763794 — **$9668\frac{22}{79}$**

1000 999 | 2341527 — **$2343\frac{870}{999}$**

120 119 | 2785857 — **$23410\frac{67}{119}$**

70 67 | 4462182 Key-word: *Add three times Q.*

$66599\frac{49}{67}$

Divide by 70	Quot.	$r + 3Q$	$= R$
446	6	26 + 18	44
442	6	22 + 18	40
401	5	51 + 15	66
668	9	38 + 27	65
652	9	22 + 27	49

60 57 | 312623 — $5484\frac{35}{57}$

100 97 | 236582 — $2438\frac{96}{97}$

90 87 | 5628435 — $64694\frac{57}{87}$

400 397 | 2847928 — $7173\frac{247}{397}$

Key-word : *Subtract once Q*, $R = r - Q$.

10 11 | 37636876 → **3421534** $\frac{2}{11}$ **30** 31 | 285670357 → **9215172** $\frac{25}{31}$

Key-word : *Subtract three times Q* ; $R = r - 3Q$.

70 73 | 6249683 → 85612 $\frac{7}{73}$ **50** 53 | 1286494 → 24273 $\frac{25}{53}$.

In the following examples, the figures in large type show where special care has to be taken with Q, so that the remainder R may neither exceed the divisor D nor be less than zero.

10 11 | 478265 → 43**478** $\frac{7}{11}$ **40** 43 | 23572864 → 548206 $\frac{6}{43}$

70 67 | 376745 → 56**23** $\frac{4}{67}$ **30** 29 | 149386 → **5151** $\frac{7}{29}$

10 9 | 32876 → 3652 $\frac{8}{9}$ **100** 96 | 327654 → **3413** $\frac{6}{96}$

70 68 | 156675 → 2304 $\frac{3}{68}$ **1000** 998 | 568976 → 570 $\frac{116}{998}$

$\frac{11}{41} = \cdot\dot{2}682\dot{9}$ **40** 41 | 11· → ·26829

$\frac{5}{17} = \cdot\dot{2}9411764$ **20** 17 | 5· → ·**29411764**

70588235

45243 farthings = £47 „ 2 „ 6$\frac{3}{4}$

1000 960 | 45243

£45 + 2043 farthings

= £45 + £2 + 123 farthings

= £47 „ 2 „ 6$\frac{3}{4}$

367234 farthings = £382 „ 10 „ 8$\frac{1}{2}$

1000 960 | 367234

£382 + 514 farthings

46236 lb. = 412 cwt. 92 lb.

3724562 lb. = 1662 tons 15 cwt. 2 lb.

110 112 | 46236 → 412 cwt. 92 lb. **110** 112 | 3724562 → 33255 cwt. 2 lb.

It is evident that sums in Long Division may be much simplified by the adoption of this method. The remainder $R = r + (d - D)Q$ may be obtained mentally

```
397 \ 284792857 / 717362 143/397
400 / 2800      \
       689
       400
       2922
       2800
        1438
        1200
         2475
         2400
           937
           800
           143
```

```
479 \ 1848479 / 3859 18/479
500 / 1500    \
       4114
       4000
        2827
        2500
         4329
         4500
           18
```

Fifth Meeting, 13th March 1903.

Dr THIRD, President, in the Chair.

Some Methods applicable to Identities and Inequalities of Symmetric Algebraic Functions of n Letters.

By R. F. MUIRHEAD, M.A., B.Sc.

The methods explained here are applicable to a large number of problems relating to the symmetric algebraic functions of n letters, and the special results here deduced from them are merely specimens to indicate some of the ways of applying these methods.

In the first section the main principle adopted is that of taking the standard form of a symmetric function to be a sum extending over all the cases of the typical term got by permuting the letters involved in all possible ways, whether they are different or not; and the main result reached is an Inequality Theorem arrived at by expressing the excess of the greater over the less in an explicitly positive form.

The method of the second section, while closely related to that of the first, is in some cases more easily applicable.

Both methods, though leading ~~to rather~~ complicated considerations in certain of their applications, are essentially elementary in their character, *i.e.*, they are based immediately on elementary ideas, and do not involve any of the more abstract conceptions of Higher Algebra.

§ 1.

Consider a term of the form $a^{\alpha}b^{\beta}c^{\gamma}\ldots l^{\lambda}$, involving n letters a, b, $c\ldots l$, each of the indices α, $\beta, \ldots \lambda$ being either a positive integer or zero.

(1) Let us denote by $\Sigma!(a^{\alpha}b^{\beta}\ldots l^{\lambda})$, the sum of all the terms that can be got by all possible permutations of a, b, $c\ldots$ in the typical

term (which is taken to involve all the letters, even if the indices of some of them are zero), while the indices α, $\beta, \ldots \lambda$ remain unchanged; so that the sum contains $n!$ terms, and is homogeneous and symmetric as to a, b, $c \ldots l$.

For example,

$$\Sigma ! (a^2 b^2 c^0) \equiv a^2 b^2 c^0 + a^2 c^2 b^0 + b^2 a^2 c^0 + b^2 c^2 a^0 + c^2 a^2 b^0 + c^2 b^2 a^0$$

and is $= 2\Sigma a^2 b^2 c^0$ or $2\Sigma a^2 b^2$, when Σ has the usual signification of summation over all *different* terms of the type $a^2 b^2$.

In general, if the indices are not all unequal, there being p of them having one common value, q of them having another common value, etc., we have

$$(2) \qquad \Sigma ! (a^\alpha b^\beta c^\gamma \ldots l^\lambda) = (p ! q ! \ldots) \times \Sigma a^\alpha b^\beta \ldots l^\lambda.$$

This notation may be extended to symmetric sums of *any* functions of a, b, $c \ldots l$ (each of which is taken to involve *all* the letters, as explained above); so that we shall have

$$(3) \qquad \Sigma ! \mathrm{F}(a, b, c \ldots l) = \mathrm{N} \Sigma \mathrm{F}(a, b, c \ldots l)$$

when F is a function of a, b, $c \ldots l$, and N is the number of times each particular *different value* of $\mathrm{F}(a, b, c \ldots l)$ occurs in the $\Sigma !$

We shall use the abbreviation

$$(4) \qquad [\alpha, \beta, \gamma \ldots \lambda] \equiv \Sigma ! (a^\alpha b^\beta \ldots l^\lambda)$$

and for brevity the commas may sometimes be omitted.

Fundamental Inequality Theorem for expressions of the type $[\alpha, \beta, \gamma \ldots \lambda]$.

The expression $\Sigma ! \{a^\beta b^\beta c^\gamma \ldots l^\lambda (a^\rho - b^\rho)(a^\sigma - b^\sigma)\}$ may be expanded into

$$\Sigma ! \{a^{\beta+\rho+\sigma} b^\beta c^\gamma \ldots l^\lambda + a^\beta b^{\beta+\rho+\sigma} c^\gamma \ldots l^\lambda$$
$$- a^{\beta+\sigma} b^{\beta+\rho} c^\gamma d^\delta \ldots l^\lambda - a^{\beta+\rho} b^{\beta+\sigma} c^\gamma \ldots l^\lambda\}$$
$$= 2\Sigma ! (a^{\beta+\rho+\sigma} b^\beta c^\gamma \ldots l^\lambda) - 2\Sigma ! (a^{\beta+\rho} b^{\beta+\sigma} c^\gamma \ldots l^\lambda).$$

Hence, writing α for $\beta+\rho+\sigma$, we have, subject to the restriction $\alpha>\beta+\sigma$,

$$(5) \qquad \Sigma\,!\,(a^{\alpha}b^{\beta}c^{\gamma}\dots l^{\lambda})-\Sigma(a^{\alpha-\sigma}b^{\beta+\sigma}c^{\gamma}\dots l^{\lambda})$$
$$=\tfrac{1}{2}\Sigma\,!\,\{a^{\beta}b^{\beta}c^{\gamma}\dots l^{\lambda}(a^{\alpha-\beta-\sigma}-b^{\alpha-\beta-\sigma})(a^{\sigma}-b^{\sigma})\}$$

whence, if $a, b, c\dots$ are all positive, we have

$$(6) \qquad \Sigma\,!\,a^{\alpha}b^{\beta}c^{\gamma}\dots l^{\lambda}>\Sigma\,!\,a^{\alpha-\sigma}b^{\beta+\sigma}c^{\gamma}\dots l^{\lambda}$$

unless $a, b, c\dots$ are all equal.

This fundamental inequality may be applied to prove the more general theorem, that if the conditions

$$(7) \qquad \alpha+\beta+\gamma\dots+\lambda=\alpha'+\beta'+\gamma'\dots+\lambda'$$

$$(8) \qquad \alpha\nless\alpha',\ \alpha+\beta\nless\alpha'+\beta',\ \alpha+\beta+\gamma\nless\alpha'+\beta'+\gamma',\ \text{etc.}$$

(one at least of the signs $\nless$ being $>$) are fulfilled then we have

$$(9) \qquad [\alpha, \beta, \gamma\dots\lambda]>[\alpha', \beta', \gamma\dots\lambda']$$

unless $a, b, c\dots$ are all equal; and to express the difference

$$[\alpha\ \beta\ \gamma\dots\lambda]-[\alpha'\ \beta'\ \gamma'\dots\lambda']$$

in a form that is essentially positive.

Take, for example, (6, 3, 2, 0, 0) − (4, 4, 1, 1, 1).

It may be written (6, 3, 2, 0, 0) − (5, 4, 2, 0, 0)
+ (5, 4, 2, 0, 0) − (4, 4, 2, 1, 0)
+ (4, 4, 2, 1, 0) − (4, 4, 1, 1, 1)

when each of the three lines is expressible by (5) as an essentially positive quantity, a, b, c, d, e being positive and not all equal.
In fact, for five letters a, b, c, d, e we have

$$\Sigma\,!\,a^6b^3c^2d^0e^0-\Sigma\,!\,a^4b^4cde$$
$$=\tfrac{1}{2}\Sigma!(a-b)(a^2-b^2)a^3b^3c^2d^0e^0+\tfrac{1}{2}\Sigma!(a-b)(a^4-b^4)c^4d^2e^0+\tfrac{1}{2}\Sigma\,!(a-b)^2c^4d^4e.$$

Similarly, to prove the general theorem (7), we must show that it is possible to insert between $[\alpha\beta\gamma\dots\lambda]$ and $[\alpha'\beta'\gamma'\dots\lambda']$ a series of quantities of the same type, but of continually descending order

as distinguished by conditions (8), such that any two successive members of the complete series formed by the two given expressions and the intermediates have a difference of the form (5). Now suppose, for example, that β is the first index in $[\alpha\beta\gamma\ldots\lambda]$ which is greater than its correspondent in $[\alpha'\beta'\gamma'\ldots\lambda']$, and that ϵ' is the first of the set α', $\beta'\ldots\lambda'$ which is greater than its correspondent ϵ. Then the quantity $[\alpha,\ \beta-1,\ \gamma,\ \delta,\ \epsilon+1,\ldots\lambda]$ is of the same degree as $[\alpha,\ \beta,\ldots\lambda]$ or $[\alpha'\ \beta',\ldots\lambda]$ but its order as tested by (8) is lower than that of $[\alpha,\ \beta,\ldots\lambda]$ but not lower than that of $(\alpha',\ \beta',\ldots\lambda']$; while the difference $[\alpha,\ \beta,\ \gamma\ldots\lambda]-[\alpha,\ \beta-1,\ \gamma,\ \delta,\ \epsilon+1,\ldots\lambda]$ is of the form (5). The expression $[\alpha,\ \beta-1,\ \gamma,\ \delta,\ \epsilon+1,\ldots\lambda]$, then, would be the first of the intermediates required to bridge the interval between $[\alpha\beta\ldots\lambda]$ and $[\alpha'\beta'\ldots\lambda']$; and it is obvious that a finite number of these will suffice.

Thus Theorem (9) is proved. We can also show that for the difference $[\alpha\beta\gamma\ldots\lambda]-[\alpha'\beta'\gamma'\ldots\lambda']$ to be a quantity necessarily positive for all *positive* values of $a, b, c\ldots$ which are not all equal, the conditions (7) and (8) are *necessary* as well as sufficient. This is done by showing that by suitably choosing the values of $a, b, c\ldots$ we can, if (7) and (8) are not satisfied, make the difference positive or negative at will.

If (7) does not hold, *i.e.*, if the quantities are of different *degree*, it is clear that by making a, b, c all equal to a sufficiently large number, the quantity of higher degree would be the greater, which if we make $a, b, c,\ldots$ all equal to a sufficiently small fraction, the quantity of lower degree would be the greater.

If (8) does not hold: if, for example, $\alpha+\beta+\gamma<\alpha'+\beta'+\gamma'$, then by making a, b, c each equal to m a sufficiently great multiple of the greatest of the remaining quantities $d, e\ldots l$, it is clear that the only important terms in $[\alpha\beta\gamma\ldots\lambda]-[\alpha'\beta'\gamma'\ldots\lambda']$ will be those having m in its highest powers, say $\mathrm{P}m^{\alpha+\beta+\gamma}-\mathrm{Q}m^{\alpha'+\beta'+\gamma'}$, which for a sufficiently large value of m is negative, $\alpha+\beta+\gamma$ being less than $\alpha'+\beta'+\gamma'$.

Thus the conditions (7) and (8) are *necessary* as well as *sufficient* to ensure that $[\alpha\beta\ldots\lambda]-[\alpha'\beta'\ldots\lambda']$ should have a value necessarily positive for all positive values of a, b, $c\ldots l$ which are not all equal.

Expression for the Excess of the Arithmetic Mean of n quantities over their Geometric Mean.

As a special case of the preceding let us take

(11) $\qquad [n-p+1,\ 1,\ 1,\ldots\ 0,\ 0]-[n-p,\ 1,\ 1,\ldots 0,\ 0]$

when in the former there are $p-1$ indices equal to 1 and $n-p$ equal to 0; and in the latter, p equal to 1 and $n-p-1$ equal to 0.

(12) The difference is $=\frac{1}{2}\Sigma\,!\,\{(a^{n-p}-b^{n-p})(a-b)c^1d^1\ldots k^0l^0\}$

where there are $p-1$ letters c, $d\ldots$ with index 1 and $n-p-1$ letters with index 0.

It may also be written

(13) $\frac{1}{2}\Sigma\,!\,\{(a-b)^2\mathrm{H}_{n-p-1}c^1d^1\ldots k^0l^0\}$

$$=(n-p-1)\,!\,(p-1)\,!\,\Sigma(a-b)^2\mathrm{H}_{n-p-1}c^1d^1\ldots\ldots$$

when $\mathrm{H}_{n-p-1}\equiv$ sum of all homogeneous products of a and b of $n-p-1$ dimensions. Note that the factor $\frac{1}{2}$ is cancelled, since $\Sigma(a-b)^2=\frac{1}{2}\Sigma\,!\,(a-b)^2$.

Writing the identity (12) for the values $1,\ 2,\ldots\overline{n-1}$ of p and combining the results, we get

(14) $[n,\ 0,\ 0\ldots]-[1,\ 1,\ 1\ldots]$

$$\begin{aligned}
=\ &(n-2)\,!\,\Sigma(a-b)(a^{n-1}-b^{n-1})\\
&+(n-3)\,!\,1\,!\,\Sigma(a-b)(a^{n-2}-b^{n-2})c\\
&+(n-4)\,!\,2\,!\,\Sigma(a-b)(a^{n-3}-b^{n-3})cd\\
&+\ldots\ldots\quad +1\,!\,(n-3)\,!\,\Sigma(a-b)(a^2-b^2)cd\ldots k\\
&\qquad\qquad +(n-2)\,!\,\Sigma(a-b)(a-b)cd\ldots kl.
\end{aligned}$$

But $[n,\ 0,\ 0\ldots]\equiv(n-1)\,!\,\Sigma a^n$, and $[1,\ 1,\ldots]\equiv n\,!\,abc\ldots kl$.

Hence, dividing by n ! we get

$$(15)\quad \frac{1}{n}\Sigma a^n - abc\ldots kl$$

$$= \frac{1}{n(n-1)}\Sigma(a-b)(a^{n-1}-b^{n-1}) + \frac{1}{n(n-1)(n-2)}\Sigma(a-b)(a^{n-2}-b^{n-2})c$$

$$+ \frac{1\,.\,2}{n(n-1)(n-2)(n-3)}\Sigma(a-b)(a^{n-3}-b^{n-3})cd + \ldots\ldots\ldots\ldots$$

$$+ \frac{1}{n(n-1)(n-2)}\Sigma(a-b)(a^2-b^2)cd\ldots k + \frac{1}{n(n-1)}\Sigma(a-b)^2 cd\ldots kl,$$

a formula which expresses the excess of the Arithmetic Mean of n positive quantities a^n, $b^n \ldots l^n$ over their Geometric Mean, in a form which shows it to be necessarily positive.

(16) It may be noted that the factor $(a-b)(a^p - b^p)$ may be written $(a-b)^2 H_{p-1}$, where $H_r \equiv$ the sum of all possible homogeneous products of a and b of the rth degree.

Consider the difference

$$(17)\qquad \frac{[6,\ 5,\ 1]}{[3,\ 3,\ 2]} - \frac{[5,\ 5,\ 3]}{[4,\ 3,\ 2]}$$

where each of the fractions is of degree 4 in $abc\ldots$.

Bringing it to a common denominator, the numerator is

$$(18)\quad [6, 5, 1] \times [4, 3, 2] - [5, 5, 3] \times [3, 3, 2]$$

$$= [10, 8, 3] + [10, 7, 4] + [9, 9, 3] + [9, 7, 5] + [8, 9, 4] + [8, 8, 5]$$

$$- [8, 8, 5] - [8, 7, 6] - [8, 8, 5] - [8, 7, 6] - [7, 8, 6] - [7, 8, 6].$$

This shows, in virtue of (9), that (16) is positive.

Under what circumstances can we prove that

$$\frac{[\alpha,\ \beta,\ \gamma\ldots]}{[p,\ q,\ r\ldots]} - \frac{[\alpha',\ \beta',\ \gamma'\ldots]}{[p',\ q',\ r'\ldots]}$$

is essentially positive for positive values of a, b, $c\ldots$ that are not all equal?

If we expand $[\alpha, \beta, \gamma\ldots] \times [p', q', r'\ldots]$ we get

$$\Sigma[\alpha + p',\ \beta + q',\ \gamma + r', \ldots],$$

where the summation extends over the n ! values got by keeping α, β, $\gamma\ldots$ in fixed order, and permuting p', q', $r'\ldots$ in all possible

ways. A similar expansion of $[\alpha', \beta', \gamma'...] \times [p, q, r...]$ can be got; and the former expansion will be necessarily greater than the latter only if we can couple each $[\alpha+p', \beta+q', \gamma+r'...]$ with a $[\alpha'+p, \beta'+q, \gamma'+r...]$ in such a way that conditions similar to (7) and (8) are satisfied for each couple. (This remark is applicable to more general cases.)

In certain cases the possibility of this will be obvious without working out the expansions; for example, when the order of $[\alpha+p', \beta+q'...]$ with $p', q',...$ in *ascending* order, is higher (as tested by (8)) than that of $[\alpha'+p, \beta'+q...]$ with $p, q,...$ in descending order of magnitude. In that case every member of the expansion $\Sigma[\alpha+p', \beta+q',...]$ will be of higher order than any member of the expansion $\Sigma[\alpha'+p, \beta'+q,...]$ using 'order' for the moment to denote the relation defined by (8).

§ 2.

Method for extending to n letters certain symmetrical Identities which may be known to hold for a restricted number of letters.

Let $a, b, c,...$ be n letters, and let $F(a, b, c...)$ or simply $F(p)$ be a function involving only p of these letters.

(19) Let $\binom{n}{p} \equiv \dfrac{n!}{p!(n-p)!}$, and $\binom{n}{0} \equiv 1$.

(20) Let $S(n, p) \equiv \Sigma F(p)$ where the summation Σ extends to all the different values of $F(p)$ got by taking different selections of the n letters to form it.

For example, if $F(2) = ab$, then $S(n, 2) \equiv ab + ac + bc + ...$
and if $F(2) = a^2b$, then $S(n, 2) \equiv a^2b + ab^2 + a^2c + ac^2 + b^2c + bc^2 + ...$

Thus S is a certain symmetric function of $a, b, c....$
Then if $n > r \nless p$, we have the following identity:

(21) $$\Sigma S(r, p) = \binom{n-p}{r-p} S(n, p)$$

where Σ here denotes the summation of all cases of $S(r, p)$ got by the $\binom{n}{r}$ combinations of r letters taken from $a, b,....$

The identity may be seen to be true by observing that each particular $F(p)$ occurs but once in $S(n, p)$, while in $\Sigma S(r, p)$ it occurs as often as there are cases of $S(r, p)$ containing it, *i.e.*, as often as its p letters can be associated with $r-p$ others out of the remaining $n-p$ letters.

Next let there be an identical relation of the form

$$AS(r, p)+BS(r, q)+\ldots\ldots=0$$

r being not less than any of the numbers $p, q\ldots$, which is known to be true for r letters, where $n>r$ and $A, B, C\ldots$ are independent of $a, b, c\ldots.$ We can by (21) generalize it so as to apply to n letters, the result being:

$$(22)\qquad \binom{n-p}{r-p}AS(n, p)+\binom{n-q}{r-q}BS(n, q)+\ldots\ldots=0.$$

In most cases it will be simplest to prove the identity first for $r=$ the greatest of the numbers $p, q\ldots.$ For example, we have for four letters, the identity

$$\Sigma a^2b^2(c-d)^2=3\Sigma a^2b^2c^2-2\Sigma a^2b^2cd.$$

Hence for n letters we get

$$\binom{n-4}{0}\Sigma a^2b^2(c-d)^2=\binom{n-3}{1}3\,.\,\Sigma a^2b^2c^2-\binom{n-4}{0}2\,.\,\Sigma a^2b^2cd$$

or
$$\Sigma a^2b^2(c-d)^2=3(n-3)\Sigma a^2b^2c^2-2\Sigma a^2b^2cd.$$

As another example, we have for six letters

$$\Sigma a^2b^2(cd-ef)^2=6\Sigma a^2b^2c^2d^2-6\Sigma a^2b^2cdef \quad - \quad - \quad \text{(i)}$$

$$\Sigma a^2b^2cd(e-f)^2=3\Sigma a^2b^2c^2de-12\Sigma a^2b^2cdef \quad - \quad - \quad \text{(ii)}$$

and for five letters

$$\Sigma a^2b^2c^2(d-e)^2=4\Sigma a^2b^2c^2d^2-2\Sigma a^2b^2c^2de$$

$\therefore$ for six letters

$$\Sigma a^2b^2c^2(d-e)^2=2\,.\,4\Sigma a^2b^2c^2d^2-2\Sigma a^2b^2c^2de \quad - \quad - \quad \text{(iii)}$$

From (i), (ii), (iii) we deduce, for six letters,

$$4\Sigma a^2b^2(cd-ef)^2=2\Sigma a^2b^2cd(e-f)^2+3\Sigma a^2b^2c^2(d-e)^2.$$

Hence by (22) we deduce for n letters $abc\ldots$

$$4\Sigma a^2b^2(cd-ef)^2=2\Sigma a^2b^2cd(e-f)^2+3(n-5)\Sigma a^2b^2c^2(d-e)^2.$$

In expanding a function such as

$$(abc+abd+acd+bcd\ldots)^2$$

in terms of $\Sigma a^2b^2c^2$, Σa^2b^2cd, etc., where the Σ applies to all the different terms of the type indicated that can be formed from n letters a, b, $c\ldots$; *First* we note that the numerical coefficients of $\Sigma a^2b^2c^2$, etc., in the required expansion do not depend on the number n, so that the coefficient

of $\Sigma a^2b^2c^2$ in $(\Sigma abc)^2$ is the same as in $(abc)^2$, and that
" Σa^2b^2cd " " " " " " " $(abc+abd+acd+bcd)^2$
" Σa^2bcde " " " " " " " $(\Sigma_5 abc)^2$

where the symbol Σ_5 means summation with reference to a, b, c, d, e only; and so on. *Secondly*, the numerical coefficient can be determined by reckoning the number of ways in which the typical product under the Σ can arise in forming the expansion.

Thus, in the above case, we have

$$(\Sigma abc)^2 = \mathrm{A}\Sigma a^2b^2c^2 + \mathrm{B}\Sigma a^2b^2cd + \mathrm{C}\Sigma a^2bcde + \mathrm{D}\Sigma abcdef.$$

Here $\mathrm{A}=1$ obviously; and $\mathrm{B}=2$ since a^2b^2cd can only arise from abc and abd, whose product, in expanding the square is multiplied by 2. Again $\mathrm{C}=6$, since a^2bcde can be separated into two factors in $\frac{1}{2}\times\binom{4}{2}$ ways, and the products of these are doubled. By similar reasoning, $\mathrm{D}=2\times\frac{1}{2}\times\binom{6}{3}=20$.

Thus $\quad (\Sigma abc)^2 = \Sigma a^2b^2c^2 + 2\Sigma a^2b^2cd + 6\Sigma a^2bcde + 20\Sigma abcdef.$

Let us study the relations between the different cases of the general function thus defined :—

(23) $(r, p, t) \equiv \Sigma\{(a^2b^2\ldots$ to $r-p-t$ factors$)\,(d.e\ldots$ to $2p$ factors$)$
$(gh\ldots$ to t factors $-\, lm\ldots$ to t factors$)^2\}$

where the summation extends to all cases that can be formed from the n letters $a, b, c, \ldots$, each case containing $r+p+t$ different letters, and being of dimensions $2r$ in the letters $a, b, c\ldots.$

(24) Note that $(r, p, 0) \equiv \Sigma(ab\ldots$ to $2p$ factors$)\,(k^2l^2\ldots$ to $r-p$ factors$)$.

It is easy to see that for $r+p+t$ letters $abc\ldots$ we have

$$(25)\quad (r, p, t)=\binom{r-p}{t}(r, p, 0)-\binom{2p+2t}{2t}\binom{2t}{t}(r, p+t, 0)$$

$$\therefore\quad (r, p+t-1, 1)=(r-p-t+1)(r, p+t-1, 0)$$
$$-\binom{2p+2t}{2}\binom{2}{1}(r, p+t, 0) \qquad (26)$$

and for $r+p+t-1$ letters

$$(r, p, t-1)=\binom{r-p}{t-1}(r, p, 0)-\binom{2p+2t-2}{2p}\binom{2t-2}{t-1}(r, p+t-1, 0).$$

Hence by (2), for $r+p+t$ letters

$$(27)\quad (r, p, t-1)=t\binom{r-p}{t-1}(r, p, 0)-\binom{2p+2t-2}{2p}\binom{2t-2}{t-1}(r, p+t-1, 0).$$

Eliminating the expressions $(r, p, 0)$, $(r, p+t, 0)$ from (25), (26), (27), we find that $(r, p+t-1, 0)$ also disappears, and we get, for $r+p+t$ letters,

$$(28)\quad (r, p, t)t\,!\,t\,!\,2p\,!=(r, p, t-1)(r-p-t+1)(t-1)\,!\,(t-1)\,!\,2p\,!$$
$$+(r, p+t-1, 1)(2p+2t-2)\,!$$

Hence, by (22), for n letters we have

$$(29)\quad (r, p, t)t\,!\,t\,!\,2p\,!$$
$$=(r, p, t-1)(n-r-p-t+1)(r-p-t+1)(t-1)\,!\,(t-1)\,!\,2p\,!$$
$$+(r, p+t-1, 1)(2p+2t-2)\,!$$

Thus any (r, p, t) can be expressed in terms of $(r, p, t-1)$ and $(r, p+t-1, 1)$. And by successive reductions of the letter t, we can finally express (r, p, t) in terms of

$$(r, p+t-1, 1), (r, p+t-2, 1), (r, p+t-3, 1)\ldots(r, p+1, 1).$$

The expression will be developed presently.

The identity (25) can by means of (22) be generalized as follows to apply to n letters.

$$(30)\quad (r, p, t)=\binom{n-r-p}{t}\binom{r-p}{t}(r, p, 0)-\binom{2p+2t}{2t}\binom{2t}{t}(r, p+t, 0).$$

To develop the expression (r, p, t) in terms of analogous expressions with $t=1$, let us for brevity put

$$\{r, p, t\}\equiv(r, p, t)\times t\,!\,t\,!\,2p\,!$$

and $\qquad a^{(m)}\equiv a(a+1)(a+2)\ldots(a+m-1).$

Then (29) becomes

$$\{r, p, t\} \quad = \{r, p+t-1, 1\} + (n-r-p-t+1)(r-p-t+1)\{r, p, t-1\}$$

$$\{r, p, t-1\} = \{r, p+t-2, 1\} + (n-r-p-t-2)(r-p-t+2)\{r, p, t-2\}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\{r, p, 2\} \quad = \{r, p+1, 1\} \quad + (n-r-p-1) \quad (r-p-t-1)\{r, p, 1\}.$$

Combining these to eliminate $\{r, p, t-1\}$, etc., we get

(31)
$$\begin{aligned}\{r, p, t\} = \{r, p+t-1, 1\} &+ (n-r-p-t+1) \quad (r-p-t+1) \quad \{r, p+t-2, 1\} \\ &+ (n-r-p-t+1)^{(2)} \, (r-p-t+1)^{(2)} \, \{r, p+t-3, 1\} \\ &+ \quad \cdots\cdots\cdots \\ &+ (n-r-p-t+1)^{(q)} \, (r-p-t+1)^{(q)} \, \{r, p+t-q-1, 1\} \\ &+ \quad \cdots\cdots\cdots \\ &+ (n-r-p-t+1)^{(t-1)}(r-p-t+1)^{(t-1)}\{r, p, 1\}\end{aligned}$$

e.g., putting $r=5$, $p=1$, $t=3$, we have

$$2!\,3!\,3!\,\Sigma a^2bc(def-ghk)^2 = 6!\,\Sigma a^2bcdefg(h-k)^2 + (n-8).2.4!\,\Sigma a^2b^2cdef(g-h)^2 + (n-8)(n-7).2.3.2!\,\Sigma a^2b^2c^2de(f-g)$$

$$\therefore \quad \Sigma a^2bc(def-ghk)^2 = 10\Sigma a^2bcdefg(h-k)^2 + \tfrac{2}{3}(n-8)\Sigma a^2b^2cdef(g-h)^2 + \tfrac{1}{6}(n-8)(n-7)\Sigma a^2b^2c^2de(f-g)^2.$$

Application to the Difference $D(n, r)$.

Let
$$D(n, r) \equiv 2\binom{n}{r-1}\binom{n}{r+1}P_r^2 - 2\binom{n}{r}^2 P_{r-1}P_{r+1}$$
where
$$P_r \equiv \Sigma abc \ldots \text{ to } r \text{ factors},$$
the summation extending over all possible products of r out of the n letters $a, b, c, \ldots$.

Thus
$$D(n, r) = \frac{2}{r(r+1)}\binom{n}{r}\binom{n}{r-1}\{r(n-r)P_r^2 - (r+1)(n-r+1)P_{r-1}P_{r+1}\}.$$

But
$$P_r^2 = (r, 0, 0) + \frac{2}{1}(r, 1, 0) + \frac{4 \cdot 3}{1 \cdot 2}(r, 2, 0) + \ldots + \binom{2p}{p}(r, p, 0) + \ldots + \binom{2r}{r}(r, r, 0)$$
where, as defined above,
$$(r, p, 0) \equiv \Sigma(abc \ldots \text{ to } 2p \text{ factors})\,(f^2g^2 \ldots \text{ to } r - 2p \text{ factors}).$$

Again $$P_{r-1}P_{r+1} = (r, 1, 0) + \binom{4}{1}(r, 2, 0) + \binom{6}{2}(r, 3, 0) + \ldots + \binom{2p}{p-1}(r, p, 0) + \ldots + \binom{2r}{r-1}(r, r, 0).$$

Hence $$D(n, r) = \frac{2}{r(r+1)}\binom{n}{r}\binom{n}{r-1}\left[r(n-r)(r, 0, 0) + \sum_{p=1}^{p=r}\left\{r(n-r)\binom{2p}{p} - (r+1)(n-r+1)\binom{2p}{p-1}\right\}(r, p, 0)\right]$$

(32) $$= \frac{2}{r(r+1)}\binom{n}{r}\binom{n}{r-1}\left[\sum_{p=0}^{p=r}\binom{2p}{p}\frac{r(n-r) - p(n-p) - p(p+1)}{p+1}(r, p, 0)\right].$$

But, putting $t = 1$ in (30), we have
$$(r, p, 1) = (n-r-p)(r-p)(r, p, 0) - (2p+2)(2p+1)(r, p+1, 0)$$
$$\therefore \binom{2p}{p}\frac{1}{p+1}(r, p, 1) = \binom{2p}{p}\frac{r(n-r) - p(n-p)}{p+1}(r, p, 0) - \binom{2p+2}{p+1}(p+1)(r, p+1, 0).$$

Hence, the expression under the symbol Σ in (32) may be written

$$\binom{2p}{p}\frac{1}{p+1}(r,\ p,\ 1)+\binom{2p+2}{p+1}(p+1)(r,\ p+1,\ 0)-\binom{2p}{p}p(r,\ p,\ 0).$$

The second and third terms cancel out in the summation, and we get

$$(33)\quad \mathrm{D}(n,\ r)=\frac{2}{r(r+1)}\binom{n}{r}\binom{n}{r-1}\sum_{p=0}^{p=r-1}\left\{\binom{2p}{p}\frac{1}{p+1}(r,\ p,\ 1)\right\}.$$

It may be noted that the coefficient $\binom{2p}{p}\cdot\frac{1}{p+1}$ or $\frac{(2p)!}{p!\,(p+1)!}$ is an integer, as can easily be shown by the aid of the identity

$$\binom{2p}{p}\div(p+1)\equiv\binom{2p+1}{p}\div(2p+1).$$

Now the dexter of (33), if $a,\ b,\ c\ldots$ are all positive and not all equal to one another, is obviously positive. Hence $\mathrm{D}(n,\ r)$ is so.

Hence we have

$$\left\{\mathrm{P}_r\div\binom{n}{r}\right\}^2>\mathrm{P}_{r-1}\mathrm{P}_{r+1}\div\left\{\binom{n}{r-1}\binom{n}{r+1}\right\}$$

an inequality proved by Euler in his *Differential Calculus*, Vol. II, 313, by the aid of the Theory of Equations. Schlömilch, who reproduces Euler's proof in his *Zeitschrift für Mathematik*, Vol. III, remarks that it would be desirable to have a proof " welche von der Natur der Sache *d.h* von combinatorischen Gründen ihren Ausgang nähme."

Note added 4th April 1903.

The formula for $\mathrm{D}(n,\ r)$ given in (33) shows that it is necessarily positive for *positive* values of $a,\ b,\ c\ldots$. But by Newton's Rule in the Theory of Equations (see Todhunter's *Theory of Equations*, Chap. XXVI) we know that $\mathrm{D}(n,\ r)$ is positive for all *real* values of $a,\ b,\ c\ldots$. It would, then, be of interest to modify the formula (33) in such a way as to make this obvious. This we can do by means of identities of the type

$$(35)\qquad \Sigma_1\{(cd\ldots\text{ to }q\text{ factors})\Sigma_2(fg\ldots\text{ to }r-q-1\text{ factors})\}^2$$

$$=\binom{r-1}{q}\phi(0)+\binom{r-2}{q}\phi(1)+\ldots+\binom{r-p-1}{q}\phi(p)+\ldots+\phi(r-q-1)$$

where $\phi(p) \equiv \binom{2p}{p}\Sigma_1\{(cd\ldots \text{ to } 2p \text{ factors}) f^2g^2\ldots \text{ to } r-p-1 \text{ factors}\}$

and Σ_1 sums all terms that can be formed from all the letters excepting a and b; while Σ_2 sums terms that can be formed from all the letters excepting a and b and the q others which have already occurred in the bracket (cd.... to q factors); and q may have any value from 0 up to $r-1$.

Denoting the sinister of (35) by G(q), and by E the coefficient of $(a-b)^2$ in the expression

$$(36)\quad D(n, r) = \frac{2}{r(r+1)}\binom{n}{r}\binom{n}{r-1}$$
$$\Sigma\left[(a-b)^2\left\{\phi(0) + \tfrac{1}{2}\phi(1) + \tfrac{1}{3}\phi(2) + \ldots + \frac{1}{r}\phi(r-1)\right\}\right]$$

which is (33) written in a slightly different form, we have

$$(37)\quad E = \frac{1}{r}G(0) + \frac{1}{r(r-1)}G(1) + \frac{1.2}{r(r-1)(r-2)}G(2) + \ldots$$
$$+ \frac{q!(r-q-1)!}{r!}G(q) + \ldots + \frac{1}{r}G(r-1).$$

Since each G is a square, and has a positive numerical coefficient in (37), it is obvious that E is essentially positive for *real* values of a, b, c..., and that D(n, r) is therefore essentially positive.

Rearranging (36) in terms of G(0), G(1), etc., we get

$$(38)\quad D(n, r) = \frac{2}{r(r+1)}\binom{n}{r}\binom{n}{r-1}$$
$$\Sigma\left[\frac{1}{r}\{(a-b)\Sigma_1 cd\ldots\}^2 + \frac{1}{r(r-1)}\{(a-b)c\,\Sigma_2 de\ldots\}^2 + \ldots\right.$$
$$+ \frac{q!(r-q-1)!}{r!}\{(a-b)(cd\ldots \text{ to } q \text{ factors})\,\Sigma_2 gh\ldots\}^2 + \ldots$$
$$\left. + \frac{1}{r}\{(a-b)(cd\ldots \text{ to } r-1 \text{ factors})\}^2\right]$$

where the typical term under Σ_2 contains sufficient letters to keep the expression homogeneous.

Construction connected with the Locus of a point at which two segments of a straight line subtend equal angles.

By R. F. Muirhead, M.A., B.Sc.

Let AB, CD be the two segments. (Fig. 49.)

Let two similar circle-segments be described on AB, CD, whose circumferences intersect in PP′.

It is obvious that ABCD passes through O the external centre of similitude of the two circles. Let OP meet the circles again in R and Q.

Then the triangles

OQC, ODP, OBR, OPA are all similar to one another

and the triangles

OQD, OCP, OPB, OAR " " " " " "

$\therefore \quad \angle DPC = \angle ARB = \angle APB.$

Again, $OD.OA = OB.OC$ so that O is a fixed point when the circles are varied.

Also, $OP^2 = OD.OA$ so that OP is a fixed length when the circles are varied.

Thus the locus of P, at which AB and CD subtend equal angles, includes the circumference of the circle of which O is the centre, and $\sqrt{OB.OC}$ the length of the radius. The locus also includes the rest of the line of which AB, CD are segments.

A totally different construction is given in the appendix to Todhunter's "Euclid," Article 54.

It may be remarked that if a circle be described on AD as diameter, and $\gamma C\gamma'$ $\beta B\beta'$ be double ordinates to this diameter, then the intersections of $\beta\gamma$, $\beta'\gamma'$; and of $\beta\gamma'$, $\beta'\gamma$ are the points in which AD meets the locus of P.

On the equation to a conic circumscribing a triangle.

By R. F. Davis, M.A.

Let $\frac{\mathrm{L}}{\alpha}+\frac{\mathrm{M}}{\beta}+\frac{\mathrm{N}}{\gamma}=0$ be the equation to a given conic circumscribing the triangle of reference ABC; p, q, r the focal chords parallel to BC, CA, AB respectively. (Fig. 50.)

Then it is well known that $\mathrm{M}\gamma+\mathrm{N}\beta=0$ is the equation to the tangent to the conic at A, so that

$$\sin\mathrm{TAB} : \sin\mathrm{TAC} = -\gamma : \beta = \mathrm{N} : \mathrm{M}.$$

Take any point P on BA produced and draw the secant PA′C′ parallel to AC.

Then
$$\begin{aligned} q : r &= \mathrm{PA}' . \mathrm{PC}' : \mathrm{PA} . \mathrm{PB} \\ &= \mathrm{PE}' . \mathrm{PC}' : \mathrm{PB}^2, \text{ if BE}' \text{ is parallel to AA}'. \end{aligned}$$

In the limit when PA′C′ moves up into coincidence with AC

$$\begin{aligned} q : r &= \mathrm{AE} . \mathrm{AC} : \mathrm{AB}^2, \text{ where BE is parallel to tangent AT} \\ &= \mathrm{AC} . \sin\mathrm{TAB} : \mathrm{AB}\sin\mathrm{TAC} \\ &= \mathrm{N}b : \mathrm{M}c ; \end{aligned}$$

$$\therefore \quad \mathrm{M}cq = \mathrm{N}br ;$$

$$\therefore \quad \mathrm{M} : \mathrm{N} = \frac{b}{q} : \frac{c}{r} .$$

Thus, by symmetry,

$$\mathrm{L} : \mathrm{M} : \mathrm{N} = \frac{a}{p} : \frac{b}{q} : \frac{c}{r} .$$

The equation to the circum-conic is therefore

$$\frac{a}{p\alpha}+\frac{b}{q\beta}+\frac{c}{r\gamma}=0.$$

The form just given is due to my old friend the Rev. T. J. Milne, and appears in the *Math. Gazette*.

The proof above is my own.

On the imaginary roots of the equation $\cos x = x$.

By T. Hugh Miller.

It has been shewn (*Proceedings of the Edin. Math. Soc.*, Vol. IX.), that this equation has only one real root, namely $x = \cdot 73908513\ldots$ and that it has an infinite number of imaginary roots of the form $A + Bi$, where A and B are given by the equations

$$A = \cos A\{e^{\tan A\sqrt{A^2-\cos^2 A}} + e^{-\tan A\sqrt{A^2-\cos^2 A}}\}/2,$$

$$B = \pm\tan A\sqrt{A^2 - \cos^2 A}. \quad - \quad - \quad - \quad - \quad - \quad \text{(i)}$$

It is, however, only half of the possible number of values that can be assigned to A and B in (i), that satisfy the equation $A + Bi = \cos(A + Bi)$, the other half satisfying the equation $A - Bi = \cos(A + Bi)$.

To discriminate between the roots we must go back to the equations from which (i) were found.

These are $$A = \cos A\frac{e^B + e^{-B}}{2} \quad - \quad - \quad - \quad - \quad \text{(1)}$$

$$B = -\sin A\frac{e^B - e^{-B}}{2} \quad - \quad - \quad - \quad - \quad - \quad \text{(2)}$$

Let the angle A be positive when it is contained between a fixed radius OC of a circle, and a movable radius OB, rotating in a left-handed way.

(1) Let A be positive.

If B is negative, $\sin A$ must be negative from equation (2), because then $e^B < e^{-B}$. Also $\cos A$ is positive by equation (1); therefore the radius OB lies in the *fourth* quadrant.

If B is positive, $\sin A$ must be negative from (2), for now $e^B > e^{-B}$. Therefore OB still lies in the fourth quadrant.

(2) Let A be negative.

Now if B be negative, $\sin A$ must be negative, but $\cos A$ is negative by equation (1). Therefore the radius OB lies in the *third* quadrant.

If B is positive, $\sin A$ and $\cos A$ must both be negative; therefore OB lies in the *third* quadrant.

Collecting these results, we find that of all the values of A and B which satisfy equations (i) only those positive values of A can be admitted which are of the form $2\pi n - C$, where n is any positive integer except 0, and C is positive and less than $\frac{\pi}{2}$; and those negative values of A which are of the form $(2n-1)\pi + C$, where n is any negative integer, including 0, and C is less than $\frac{\pi}{2}$.

In each case there are two roots for any value of n, namely those formed by giving B the positive and the negative sign.

Thus for positive values of A

$$\cos\{(2n\pi - C) \pm Bi\} = \cos C\frac{e^{B}+e^{-B}}{2} \pm i \,.\, \sin C \,.\, \frac{e^{B}-e^{-B}}{2},$$

and a similar equation for the negative values of A.

When A and B become large, the positive values of A give as an approximate value of x

$$\left(2n\pi - \frac{\log 4n\pi}{2n\pi}\right) \pm i\left[\log 4n\pi - \left(\frac{\log 4n\pi}{2n\pi}\right)^2\right].$$

The smallest positive value of A that satisfies equations (i) with the above conditions is 5·86956, and the corresponding roots of the equation are

$$5{\cdot}86956 \pm i2{\cdot}5449.$$

The next in order are $12{\cdot}30856 \pm i3{\cdot}2349$,

$$18{\cdot}657 \quad \pm i3{\cdot}6191.$$

Corresponding to the numerically smallest value of A, is the root

$$-2{\cdot}48714 \pm i1{\cdot}8093.$$

FIGURE 51.

To interpret these results; let O be the centre of a circle of unit radius, let OA be the initial line cutting the circle in C. Draw a radius OB, so that the ratio of the area COB to half the square on the radius, shall equal A.

Next describe a rectangular hyperbola having OB for transverse axis. Take a point P on the hyperbola, joining OP, so that the ratio of the area BOP to half the square on the radius of the circle is B.

From P draw PM, PN perpendicular to the axis of the hyperbola and to the initial line respectively; draw MR perpendicular to the initial line and produce it to Q making RQ equal to NR. Join OQ; OQ is the cosine of $A + Bi$.

$$\text{For}\quad OM = \frac{e^B + e^{-B}}{2} = \cos iB,$$

$$\text{and}\quad PM = \frac{e^B - e^{-B}}{2} = \frac{\sin iB}{i},$$

$$\therefore\quad \sin iB = iPM.$$

$$\begin{aligned}\therefore\quad \cos(A + Bi) &= \cos A \cos iB - \sin A \sin iB \\ &= \cos A \,.\, OM - \sin A \,.\, iPM. \\ &= OR - i \,.\, NR. \\ &= OQ.\end{aligned}$$

To construct the diagram; the angle BOC is obviously equal to A. Let the angle POB be called ψ, then

$$\text{area}\quad POB = \tfrac{1}{2}(OC)^2 \log\sqrt{\frac{1 + \tan\psi}{1 - \tan\psi}}.$$

$$\therefore\quad B = \log\sqrt{\frac{1 + \tan\psi}{1 - \tan\psi}};$$

$$\therefore\quad \frac{e^B - e^{-B}}{e^B + e^{-B}} = \tan\psi.$$

The limits of ψ are $\frac{\pi}{4}$ and $-\frac{\pi}{4}$.

Sixth Meeting, 8th May, 1903.

Mr CHARLES TWEEDIE in the Chair.

On the convergents to a recurring continued fraction, with application to finding integral solutions of the equation $x^2 - Cy^2 = (-1)^n D_n$.

By ALEXANDER HOLM, M.A.

1. Let $$\frac{E+\sqrt{C}}{D} = a_1 + \frac{1}{a_2+}\cdots\cdots\frac{1}{a_k+}\;\frac{1}{\dot{a}_1+}\;\frac{1}{a_2+}\cdots\cdots\frac{1}{\dot{a}_c+}\cdots\cdots \quad (1)$$

where D and E are integers, and C a positive integer not a perfect square, k being the number of partial quotients in the non-recurring part of the continued fraction, and c the number in the cycle,

and let $$\frac{E'+\sqrt{C}}{D'} = \dot{a}_1 + \frac{1}{a_2+}\cdots\cdots\frac{1}{\dot{a}_c+}\cdots\cdots \qquad - \quad - \quad (2)$$

D′ and E′ being integers.

If $\frac{p_s}{q_s}$ and $\frac{P_s}{Q_s}$ are the sth convergents to the continued fractions (1) and (2),

then $$\frac{p_{k+mc+s}}{q_{k+mc+s}} = a_1 + \frac{1}{a_2+}\cdots\frac{1}{a_k+}\;\frac{1}{a_1+}\;\frac{1}{a_2+}\cdots\frac{1}{a_{mc}+}\;\frac{1}{a_1+}\;\frac{1}{a_2+}\cdots\frac{1}{a_s}$$

$$= a_1 + \frac{1}{a_2+}\cdots\frac{1}{a_k+}\;\frac{1}{a_1+}\;\frac{1}{a_2+}\cdots\frac{1}{a_{mc}+}\;\frac{1}{\frac{P_s}{Q_s}}$$

$$= \frac{\frac{P_s}{Q_s}p_{k+mc} + p_{k+mc-1}}{\frac{P_s}{Q_s}q_{k+mc} + q_{k+mc-1}}.$$

Put $\qquad r=k+mc\,;$

$$\therefore \quad \frac{p_{r+s}}{q_{r+s}}=\frac{p_r\mathrm{P}_s+p_{r-1}\mathrm{Q}_s}{q_r\mathrm{P}_s+q_{r-1}\mathrm{Q}_s}.$$

The numerator of this fraction is prime to the denominator ; *

$$\therefore \quad \left.\begin{aligned} p_{r+s}&=p_r\mathrm{P}_s+p_{r-1}\mathrm{Q}_s \\ q_{r+s}&=q_r\mathrm{P}_s+q_{r-1}\mathrm{Q}_s \end{aligned}\right\} \quad \text{- - - - - - (3).}$$

Again

$$\frac{\mathrm{E}+\sqrt{\mathrm{C}}}{\mathrm{D}}=a_1+\frac{1}{a_2+}\cdots\frac{1}{a_k+}\frac{1}{a_1+}\frac{1}{a_2+}\cdots\frac{1}{a_{mc}+}\frac{1}{a_1+}\frac{1}{a_2+}\cdots\frac{1}{a_c+}\cdots$$

$$=a_1+\frac{1}{a_2+}\cdots\frac{1}{a_k+}\frac{1}{a_1+}\frac{1}{a_2+}\cdots\frac{1}{a_{mc}+}\frac{1}{\dfrac{\mathrm{E}'+\sqrt{\mathrm{C}}}{\mathrm{D}'}}$$

$$=\frac{\dfrac{\mathrm{E}'+\sqrt{\mathrm{C}}}{\mathrm{D}'}p_{k+mc}+p_{k+mc-1}}{\dfrac{\mathrm{E}'+\sqrt{\mathrm{C}}}{\mathrm{D}'}q_{k+mc}+q_{k+mc-1}};$$

$$\therefore \quad \frac{\mathrm{E}+\sqrt{\mathrm{C}}}{\mathrm{D}}=\frac{\dfrac{\mathrm{E}'+\sqrt{\mathrm{C}}}{\mathrm{D}'}p_r+p_{r-1}}{\dfrac{\mathrm{E}'+\sqrt{\mathrm{C}}}{\mathrm{D}'}q_r+q_{r-1}}.$$

From this eliminate p_{r-1} and q_{r-1} by means of (3).

$$\therefore \quad \frac{\mathrm{E}+\sqrt{\mathrm{C}}}{\mathrm{D}}=\frac{\dfrac{\mathrm{E}'+\sqrt{\mathrm{C}}}{\mathrm{D}'}p_r\mathrm{Q}_s+p_{r+s}-p_r\mathrm{P}_s}{\dfrac{\mathrm{E}'+\sqrt{\mathrm{C}}}{\mathrm{D}'}q_r\mathrm{Q}_s+q_{r+s}-q_r\mathrm{P}_s}$$

$$=\frac{p_{r+s}-p_r\left(\mathrm{P}_s-\dfrac{\mathrm{E}'+\sqrt{\mathrm{C}}}{\mathrm{D}'}\mathrm{Q}_s\right)}{q_{r+s}-q_r\left(\mathrm{P}_s-\dfrac{\mathrm{E}'+\sqrt{\mathrm{C}}}{\mathrm{D}'}\mathrm{Q}_s\right)};$$

$$\therefore \quad p_{r+s}-\frac{\mathrm{E}+\sqrt{\mathrm{C}}}{\mathrm{D}}q_{r+s}=\left(p_r-\frac{\mathrm{E}+\sqrt{\mathrm{C}}}{\mathrm{D}}q_r\right)\left(\mathrm{P}_s-\frac{\mathrm{E}'+\sqrt{\mathrm{C}}}{\mathrm{D}'}\mathrm{Q}_s\right).$$

The above is similar to Serret, pp. 60–61, except that the period

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* See Serret, *Alg. Sup.*, 4me éd. t. i., p. 61.

has been made to begin at the first quotient of the cycle instead of at any quotient.

Now $r = k + mc$;

$$\therefore\ p_{k+mc+s} - \frac{E+\sqrt{C}}{D} q_{k+mc+s}$$

$$= \left(p_{k+mc} - \frac{E+\sqrt{C}}{D} q_{k+mc}\right)\left(P_s - \frac{E'+\sqrt{C}}{D'} Q_s\right) \quad (4)$$

and in particular when $m = 0$,

$$p_{k+s} - \frac{E+\sqrt{C}}{D} q_{k+s} = \left(p_k - \frac{E+\sqrt{C}}{D} q_k\right)\left(P_s - \frac{E'+\sqrt{C}}{D'} Q_s\right) \quad (5).$$

Then let $s = c$ in (4).

$$\therefore\ p_{k+(m+1)c} - \frac{E+\sqrt{C}}{D} q_{k+(m+1)c}$$

$$= \left(p_{k+mc} - \frac{E+\sqrt{C}}{D} q_{k+mc}\right)\left(P_c - \frac{E'+\sqrt{C}}{D'} Q_c\right)$$

$$= \left(p_{k+(m-1)c} - \frac{E+\sqrt{C}}{D} q_{k+(m-1)c}\right)\left(P_c - \frac{E'+\sqrt{C}}{D'} Q_c\right)^2 \text{ similarly.}$$

....

$$= \left(p_k - \frac{E+\sqrt{C}}{D} q_k\right)\left(P_c - \frac{E'+\sqrt{C}}{D'} Q_c\right)^{m+1};$$

or writing m for $m+1$,

$$p_{k+mc} - \frac{E+\sqrt{C}}{D} q_{k+mc} = \left(p_k - \frac{E+\sqrt{C}}{D} q_k\right)\left(P_c - \frac{E'+\sqrt{C}}{D'} Q_c\right)^m \quad (6).$$

Substitute this in (4).

$$\therefore\ p_{k+mc+s} - \frac{E+\sqrt{C}}{D} q_{k+mc+s}$$

$$= \left(p_k - \frac{E+\sqrt{C}}{D} q_k\right)\left(P_s - \frac{E'+\sqrt{C}}{D'} Q_s\right)\left(P_c - \frac{E'+\sqrt{C}}{D'} Q_c\right)^m$$

$$= \left(p_{k+s} - \frac{E+\sqrt{C}}{D} q_{k+s}\right)\left(P_c - \frac{E'+\sqrt{C}}{D'} Q_c\right)^m \quad \text{by (5).}$$

Now put $k+s=n$ so that $n \nless k$;

$$\therefore\ p_{mc+n} - \frac{E+\sqrt{C}}{D} q_{mc+n} = \left(p_n - \frac{E+\sqrt{C}}{D} q_n\right)\left(P_c - \frac{E'+\sqrt{C}}{D'} Q_c\right)^m \quad (7)$$

where $m=1, 2, 3, \ldots\ldots$ and $n \nless k$.

In this way we can dispense with the tedious demonstration of Serret, § 28, to prove that what corresponds to $P_c - \frac{E'+\sqrt{C}}{D'} Q_c$ in (7) is constant, no matter at what quotient of the cycle the period is made to begin.

Example: $\frac{123+\sqrt{37}}{28} = 4 + \frac{1}{1+}\ \frac{1}{1+}\ \frac{1}{1+}\ \frac{1}{1+}\ \frac{1}{3+}\ \frac{1}{2+}\ \ldots\ldots$

$c=3$, and the first five convergents are $\frac{4}{1}, \frac{5}{1}, \frac{9}{2}, \frac{14}{3}, \frac{23}{5}$,

and $\frac{3+\sqrt{37}}{7} = 1 + \frac{1}{3+}\ \frac{1}{2+}\ \ldots\ldots$

the first three convergents being $\frac{1}{1}, \frac{4}{3}, \frac{9}{7}$.

Putting $n=5$, $m=2$ in (7) we have

$$p_{11} - \frac{123+\sqrt{37}}{28} q_{11} = \left(23 - \frac{123+\sqrt{37}}{28} \times 5\right)\left(9 - \frac{3+\sqrt{37}}{7} \times 7\right)^2$$

$$= \frac{4337 - 713\sqrt{37}}{28}$$

$$\therefore\ q_{11} = 713$$

$$\text{and}\ \ p_{11} = \frac{123}{28} \times 713 + \frac{4337}{28} = 3287.$$

2. Particular case of a pure recurring continued fraction.

Let $\frac{E'+\sqrt{C}}{D'} = a_1 + \frac{1}{a_2+}\ \ldots\ldots \frac{1}{a_c+}\ \ldots\ldots$

Here $D = D'$, $E = E'$,

$p_s = P_s$, $q_s = Q_s$;

$\therefore$ by (7) we have

$$P_{mc+n} - \frac{E'+\sqrt{C}}{D'} Q_{mc+n} = \left(P_n - \frac{E'+\sqrt{C}}{D'} Q_n\right)\left(P_c - \frac{E'+\sqrt{C}}{D'} Q_c\right)^m \quad (8).$$

In particular let $n=c$, and then write m for $m+1$.

$$\therefore \quad P_{mc}-\frac{E'+\sqrt{C}}{D'}Q_{mc}=\left(P_c-\frac{E'+\sqrt{C}}{D'}Q_c\right)^m \qquad (9).$$

3. To connect the convergents of the continued fraction representing $-\dfrac{E'-\sqrt{C}}{D'}$ with those of the continued fraction representing $\dfrac{E'+\sqrt{C}}{D'}$, when the latter is a pure recurring continued fraction.

Let $$x=\frac{E'+\sqrt{C}}{D'}=\dot{a}_1+\frac{1}{a_2+}\cdots\cdots\frac{1}{\dot{a}_c+}\cdots\cdots$$

$$\therefore \quad x=a_1+\frac{1}{a_2+}\cdots\cdots\frac{1}{a_{mc}+}\frac{1}{x};$$

$$\therefore \quad x=\frac{xP_{mc}+P_{mc-1}}{xQ_{mc}+Q_{mc-1}};$$

$$\therefore \quad Q_{mc}x^2-(P_{mc}-Q_{mc-1})x-P_{mc-1}=0 \qquad (10)$$

$\therefore \quad \dfrac{E'+\sqrt{C}}{D'}$ is one of the roots of this quadratic equation.

The other root $$\frac{E'-\sqrt{C}}{D'}=-\frac{1}{\dot{a}_c+}\frac{1}{a_{c-1}+}\cdots\cdots\frac{1}{a_2+}\frac{1}{\dot{a}_1+}\cdots\cdots \qquad (11).$$

(See Serret, p. 49.)

$$\therefore \quad -\frac{E'-\sqrt{C}}{D'}=\frac{1}{\dot{a}_{mc}+}\frac{1}{a_{mc-1}+}\cdots\cdots\frac{1}{a_2+}\frac{1}{\dot{a}_1+}\cdots\cdots$$

Let $\dfrac{P'_r}{Q'_r}$ be the rth convergent.

Then if $n<mc$, $$\frac{P'_{mc-n}}{Q'_{mc-n}}=\frac{1}{a_{mc}+}\frac{1}{a_{mc-1}+}\cdots\cdots\frac{1}{a_{n+2}+}\frac{1}{a_{n+1}};$$

$$\therefore \quad \frac{Q'_{mc-n}}{Q'_{mc-n-1}}=a_{n+1}+\frac{1}{a_{n+2}+}\cdots\cdots\cdots\cdots\frac{1}{a_{mc}},$$

and $$\frac{P'_{mc-n}}{P'_{mc-n-1}}=a_{n+1}+\frac{1}{a_{n+2}+}\cdots\cdots\cdots\cdots\frac{1}{a_{mc-1}}.$$

Now
$$\frac{P_{mc}}{Q_{mc}} = a_1 + \frac{1}{a_2 +} \cdots \frac{1}{a_n +} \frac{1}{a_{n+1} +} \frac{1}{a_{n+2} +} \cdots\cdots \frac{1}{a_{mc}}$$

$$= a_1 + \frac{1}{a_2 +} \cdots\cdots \frac{1}{a_n + \cfrac{1}{\cfrac{Q'_{mc-n}}{Q'_{mc-n-1}}}}$$

$$= \frac{\dfrac{Q'_{mc-n}}{Q'_{mc-n-1}} P_n + P_{n-1}}{\dfrac{Q'_{mc-n}}{Q'_{mc-n-1}} Q_n + Q_{n-1}}$$

$$= \frac{P_n Q'_{mc-n} + P_{n-1} Q'_{mc-n-1}}{Q_n Q'_{mc-n} + Q_{n-1} Q'_{mc-n-1}}.$$

The numerator is prime to the denominator.

$$\therefore \quad P_n Q'_{mc-n} + P_{n-1} Q'_{mc-n-1} = P_{mc}$$

$$\text{and} \quad Q_n Q'_{mc-n} + Q_{n-1} Q'_{mc-n-1} = Q_{mc}.$$

Eliminate Q'_{mc-n}.

$$\therefore \quad (P_n Q_{n-1} - P_{n-1} Q_n) Q'_{mc-n-1} = P_n Q_{mc} - Q_n P_{mc};$$

$$\therefore \quad (-1)^n Q'_{mc-n-1} = P_n Q_{mc} - Q_n P_{mc}.$$

Again
$$\frac{P_{mc-1}}{Q_{mc-1}} = a_1 + \frac{1}{a_2 +} \cdots\cdots \frac{1}{a_n +} \frac{1}{a_{n+1} +} \frac{1}{a_{n+2} +} \cdots\cdots \frac{1}{a_{mc-1}}$$

$$= a_1 + \frac{1}{a_2 +} \cdots\cdots\cdots \frac{1}{a_n + \cfrac{1}{\cfrac{P'_{mc-n}}{P'_{mc-n-1}}}},$$

and in the same way as above we find

$$(-1)^n P'_{mc-n-1} = P_n Q_{mc-1} - Q_n P_{mc-1};$$

$$\therefore \quad (-1)^n \left(P'_{mc-n-1} + \frac{E' + \sqrt{C}}{D'} Q'_{mc-n-1} \right)$$

$$= P_n \left(Q_{mc-1} + \frac{E' + \sqrt{C}}{D'} Q_{mc} \right) - Q_n \left(P_{mc-1} + \frac{E' + \sqrt{C}}{D'} P_{mc} \right).$$

Now

$$\frac{E' + \sqrt{C}}{D'} + \frac{E' - \sqrt{C}}{D'} = \text{sum of roots of the quadratic (10)} = \frac{P_{mc} - Q_{mc-1}}{Q_{mc}};$$

$$\therefore \quad Q_{mc-1} + \frac{E' + \sqrt{C}}{D'} Q_{mc} = P_{mc} - \frac{E' - \sqrt{C}}{D'} Q_{mc}$$

and $\dfrac{E'+\sqrt{C}}{D'} \cdot \dfrac{E'-\sqrt{C}}{D'}$ = product of roots of the quadratic $= -\dfrac{P_{mc-1}}{Q_{mc}}$;

$$\therefore \quad P_{mc-1} + \frac{E'+\sqrt{C}}{D'} P_{mc} = \frac{E'+\sqrt{C}}{D'}\left(P_{mc} - \frac{E'-\sqrt{C}}{D'} Q_{mc}\right);$$

$$\therefore \quad (-1)^n\left(P'_{mc-n-1} + \frac{E'+\sqrt{C}}{D'} Q'_{mc-n-1}\right)$$

$$= \left(P_n - \frac{E'+\sqrt{C}}{D'} Q_n\right)\left(P_{mc} - \frac{E'-\sqrt{C}}{D'} Q_{mc}\right).$$

Again $\left(P_{mc} - \dfrac{E'+\sqrt{C}}{D'} Q_{mc}\right)\left(P_{mc} - \dfrac{E'-\sqrt{C}}{D'} Q_{mc}\right)$

$$= P_{mc}^2 - \left(\frac{E'+\sqrt{C}}{D'} + \frac{E'-\sqrt{C}}{D'}\right) P_{mc} Q_{mc} + \frac{E'+\sqrt{C}}{D'} \cdot \frac{E'-\sqrt{C}}{D'} Q_{mc}^2$$

$$= P_{mc}^2 - \frac{P_{mc} - Q_{mc-1}}{Q_{mc}} \cdot P_{mc} Q_{mc} - \frac{P_{mc-1}}{Q_{mc}} \cdot Q_{mc}^2$$

$$= P_{mc} Q_{mc-1} - P_{mc-1} Q_{mc}$$

$$= (-1)^{mc};$$

$\therefore$ by (9) we have

$$\left(P_c - \frac{E'+\sqrt{C}}{D'} Q_c\right)^m \left(P_{mc} - \frac{E'-\sqrt{C}}{D'} Q_{mc}\right) = (-1)^{mc};$$

$$\therefore \quad P_{mc} - \frac{E'-\sqrt{C}}{D'} Q_{mc} = (-1)^{mc}\left(P_c - \frac{E'+\sqrt{C}}{D'} Q_c\right)^{-m};$$

$$\therefore \quad P'_{mc-n-1} + \frac{E'+\sqrt{C}}{D'} Q'_{mc-n-1}$$

$$= (-1)^{mc+n}\left(P_n - \frac{E'+\sqrt{C}}{D'} Q_n\right)\left(P_c - \frac{E'+\sqrt{C}}{D'} Q_c\right)^{-m} \quad (12).$$

The relations connecting the convergents of $-\dfrac{E'-\sqrt{C}}{D'}$ with those of $\dfrac{E'+\sqrt{C}}{D'}$ are thus established directly.

Comparing (12) with (8) we see that the power of $P_c - \dfrac{E'+\sqrt{C}}{D'} Q_c$ is $-m$ instead of m.

Example, $$\frac{3+\sqrt{37}}{7}=1+\frac{1}{3+}\;\frac{1}{\dot{2}+}\;\dots\dots$$

$c=3$, and the first three convergents are $\frac{1}{1}$, $\frac{4}{3}$, $\frac{9}{7}$.

Taking $n=2$, $m=3$ in (12) we have

$$P'_6+\frac{3+\sqrt{37}}{7}Q'_6=(-1)^{11}\left(4-\frac{3+\sqrt{37}}{7}\times 3\right)\left(9-\frac{3+\sqrt{37}}{7}\times 7\right)^{-3}$$

$$=\frac{663+109\sqrt{37}}{7};$$

$$\therefore\quad Q'_6=109;$$

$$\therefore\quad P'_6\quad=-\frac{3}{7}\times 109+\frac{663}{7}=48.$$

Thus $\frac{48}{109}$ is the sixth convergent to

$$-\frac{3-\sqrt{37}}{7}=\frac{1}{\dot{2}+}\;\frac{1}{3+}\;\frac{1}{\dot{1}+}\;\dots\dots.$$

4. For a pure quadratic surd, let

$$\frac{\sqrt{C}}{D}=a+\frac{1}{\dot{a}_1+}\;\frac{1}{a_2+}\;\dots\dots\;\frac{1}{a_{c-1}+}\;\frac{1}{2\dot{a}+}\;\dots\dots$$

$\frac{p_r}{q_r}$ being the rth convergent.

Then $$\frac{aD+\sqrt{C}}{D}=a+\frac{\sqrt{C}}{D}=\dot{2a}+\frac{1}{a_1+}\;\frac{1}{a_2+}\;\dots\dots\frac{1}{\dot{a}_{c-1}+}\;\dots\dots$$

$\therefore\ \frac{aD+\sqrt{C}}{D}$ is represented by a pure recurring continued fraction,

and if $\frac{P_r}{Q_r}$ is the rth convergent, by (8) we have

$$P_{mc+n}-\frac{aD+\sqrt{C}}{D}Q_{mc+n}=\left(P_n-\frac{aD+\sqrt{C}}{D}Q_n\right)\left(P_c-\frac{aD+\sqrt{C}}{D}Q_c\right)^m.$$

Now $$\frac{P_r}{Q_r}=2a+\frac{1}{a_1+}\;\frac{1}{a_2+}\;\dots\dots\frac{1}{a_{r-1}}=a+\frac{p_r}{q_r}=\frac{p_r+aq_r}{q_r}.$$

The numerator is prime to the denominator.

$$\therefore \quad P_r = p_r + aq_r \text{ and } Q_r = q_r;$$

$$\therefore \quad p_{mc+n} + aq_{mc+n} - \frac{aD + \sqrt{C}}{D} q_{mc+n}$$

$$= \left(p_n + aq_n - \frac{aD + \sqrt{C}}{D} q_n\right)\left(p_c + aq_c - \frac{aD + \sqrt{C}}{D} q_c\right)^m;$$

$$\therefore \quad p_{mc+n} - \frac{\sqrt{C}}{D} q_{mc+n} = \left(p_n - \frac{\sqrt{C}}{D} q_n\right)\left(p_c - \frac{\sqrt{C}}{D} q_c\right)^m \quad . \quad (13).$$

Example $\qquad \dfrac{\sqrt{80}}{5} = 1 + \dfrac{1}{\dot{1}+}\ \dfrac{1}{3+}\ \dfrac{1}{1+}\ \dfrac{1}{\dot{2}+} \ \ldots\ldots$

$c = 4$, and the first four convergents are $\dfrac{1}{1}, \dfrac{2}{1}, \dfrac{7}{4}, \dfrac{9}{5}$.

Taking $n = 3$ and $m = 2$ in (13) we have

$$p_{11} - \frac{\sqrt{80}}{5} q_{11} = \left(7 - \frac{\sqrt{80}}{5} \times 4\right)\left(9 - \frac{\sqrt{80}}{5} \times 5\right)^2 = 2279 - \frac{1274\sqrt{80}}{5};$$

$$\therefore \quad p_{11} = 2279 \text{ and } q_{11} = 1274.$$

Again, since $\dfrac{aD + \sqrt{C}}{D} = 2a + \dfrac{1}{\dot{a}_1+}\ \dfrac{1}{a_2+} \ldots\ldots \dfrac{1}{\dot{a}_{mc-1}+} \ldots\ldots$

$$\therefore \text{ by (11)} \quad -\frac{aD - \sqrt{C}}{D} = \frac{1}{\dot{a}_{mc-1}+}\ \frac{1}{a_{mc-2}+} \ldots \frac{1}{a_2+}\ \frac{1}{a_1+}\ \frac{1}{2\dot{a}+} \ldots \quad (14)$$

$$\therefore \quad \frac{\sqrt{C}}{D} = a + \frac{1}{\dot{a}_{mc-1}+}\ \frac{1}{a_{mc-2}+} \ldots \frac{1}{a_2+}\ \frac{1}{a_1+}\ \frac{1}{2\dot{a}+} \ldots \quad (15).$$

Let $\dfrac{p_r}{q_r}$ and $\dfrac{P'_r}{Q'_r}$ be the rth convergents to the continued fractions (15) and (14).

Then by (12) we have

$$P'_{mc-n-1} + \frac{aD + \sqrt{C}}{D} Q'_{mc-n-1}$$

$$= (-1)^{mc+n}\left(P_n - \frac{aD + \sqrt{C}}{D} Q_n\right)\left(P_c - \frac{aD + \sqrt{C}}{D} Q_c\right)^{-m}.$$

Now $\dfrac{p_{r+1}}{q_{r+1}} = a + \dfrac{1}{a_{mc-1}+}\,\dfrac{1}{a_{mc-2}+}\,\dots\dots\,\dfrac{1}{a_{mc-r}} = a + \dfrac{P'_r}{Q'_r} = \dfrac{P'_r + aQ'_r}{Q'_r}$.

The numerator is prime to the denominator.

$$\therefore \quad Q'_r = q_{r+1} \text{ and } P'_r + aQ'_r = p_{r+1}\,;$$

$$\therefore \quad P'_r = p_{r+1} - aq_{r+1}\,;$$

$$\therefore \quad p_{mc-n} - aq_{mc-n} + \frac{aD+\sqrt{C}}{D}q_{mc-n}$$

$$= (-1)^{mc+n}\left(p_n + aq_n - \frac{aD+\sqrt{C}}{D}q_n\right)\left(p_c + aq_c - \frac{aD+\sqrt{C}}{D}q_c\right)^{-m};$$

$$\therefore \quad p_{mc-n} + \frac{\sqrt{C}}{D}q_{mc-n} = (-1)^{mc+n}\left(p_n - \frac{\sqrt{C}}{D}q_n\right)\left(p_c - \frac{\sqrt{C}}{D}q_c\right)^{-m} \quad (16).$$

Example $\qquad \dfrac{\sqrt{80}}{5} = 1 + \dfrac{1}{\dot{1}+}\,\dfrac{1}{3+}\,\dfrac{1}{1+}\,\dfrac{1}{\dot{2}+}\,\dots\dots$.

Taking $n=3$, $m=2$ in (16) we find

$$p_5 + \frac{\sqrt{80}}{5}q_5 = (-1)^{11}\left(7 - \frac{\sqrt{80}}{5}\times 4\right)\left(9 - \frac{\sqrt{80}}{5}\times 5\right)^{-2} = 25 + \frac{14\sqrt{80}}{5};$$

$$\therefore \quad p_5 = 25 \text{ and } q_5 = 14.$$

The formulae (13) and (16) could be obtained independently, and on account of their importance in the solution of indeterminate equations of the second degree, it might be advisable to prove them directly.

5. To find integral solutions of $x^2 - Cy^2 = (-1)^n D_n$, where D_n is the $(n+1)$th divisor in the development of $\sqrt{C}$ as a continued fraction.

(i) when c, the number of quotients in the cycle of $\sqrt{C}$ is even.

Taking $D = 1$ in (13) we have

$$p_{mc+n} - \sqrt{C}.q_{mc+n} = (p_n - \sqrt{C}.q_n)(p_c - \sqrt{C}.q_c)^m.$$

Now p_{mc+n}, q_{mc+n}, p_n, q_n, p_c, q_c are all rational, whereas $\sqrt{C}$ is irrational, hence we may change the sign of $\sqrt{C}$.

$$\therefore \quad p_{mc+n} + \sqrt{C}.q_{mc+n} = (p_n + \sqrt{C}.q_n)(p_c + \sqrt{C}.q_c)^m.$$

On multiplying we obtain

$$p_{mc+n}^2 - Cq_{mc+n}^2 = (p_n^2 - Cq_n^2)(p_c^2 - Cq_c^2)^m$$
$$= (-1)^n D_n \{(-1)^c D_c\}^m$$
$$= (-1)^n D_n,$$

since c is even, and $D_c = 1$.

Similarly from (16) we get

$$p_{mc-n}^2 - Cq_{mc-n}^2 = (-1)^n D_n.$$

$\therefore$ integral solutions of $x^2 - Cy^2 = (-1)^n D_n$

$$\text{are} \quad \left.\begin{aligned} x &= \pm p_{mc+n} \\ y &= \pm q_{mc+n} \end{aligned}\right\} \quad \text{and} \quad \left.\begin{aligned} x &= \pm p_{mc-n} \\ y &= \mp q_{mc-n} \end{aligned}\right\};$$

$$\therefore \quad x - y\sqrt{C} = \pm(p_{mc+n} - \sqrt{C}\,.\,q_{mc+n}) \text{ or } \pm(p_{mc-n} + \sqrt{C}\,.\,q_{mc-n})$$
$$= \pm(p_n - \sqrt{C}\,.\,q_n)(p_c - \sqrt{C}\,.\,q_c)^{\pm m} \qquad \text{by (13) and (16).}$$

$\therefore$ integral solutions of $x^2 - Cy^2 = (-1)^n D_n$ are furnished by

$$x - y\sqrt{C} = \pm(p_n - q_n\sqrt{C})(p_c - q_c\sqrt{C})^m \qquad (17)$$

where m is zero, or any integer positive or negative.

Example: Find positive integral solutions of $x^2 - 7y^2 = -3$.

We have $\qquad \sqrt{7} = 2 + \dfrac{1}{\dot{1}+}\,\dfrac{1}{1+}\,\dfrac{1}{1+}\,\dfrac{1}{\dot{4}+}\,\cdots\cdots$

$c = 4$, and the first four convergents are $\dfrac{2}{1}, \dfrac{3}{1}, \dfrac{5}{2}, \dfrac{8}{3}$.

When $n = 1$, $\qquad p_n^2 - 7q_n^2 = 2^2 - 7 \times 1^2 = -3$;

$\therefore$ by (17) we have

$$x - y\sqrt{7} = \pm(2 - \sqrt{7})(8 - 3\sqrt{7})^m.$$

Taking $\qquad m = 0, \; -1, \; +1, \; -2, \; +2 \ldots\ldots$

we find $\qquad \left.\begin{aligned} x &= 2 \\ y &= 1 \end{aligned}\right\} \left.\begin{aligned} 5 \\ 2 \end{aligned}\right\} \left.\begin{aligned} 37 \\ 14 \end{aligned}\right\} \left.\begin{aligned} 82 \\ 31 \end{aligned}\right\} \left.\begin{aligned} 590 \\ 223 \end{aligned}\right\} \ldots\ldots$

(ii) When c, the number of partial quotients in the cycle of $\sqrt{C}$ is odd.

$(-1)^c = -1$, but if m is even $\{(-1)^c\}^m = +1$;

and proceeding in the same way as above we see that integral solutions of $x^2 - Cy^2 = (-1)^n D_n$ are given by

$$x - y\sqrt{C} = \pm(p_n - q_n\sqrt{C})(p_c - q_c\sqrt{C})^{2m} \quad - \quad - \quad (18)$$

where m is zero, or any integer positive or negative.

Example: Find positive integral solutions of $x^2 - 13y^2 = +3$.

We have $\sqrt{13} = 3 + \frac{1}{1+}\ \frac{1}{1+}\ \frac{1}{1+}\ \frac{1}{1+}\ \frac{1}{6+} \cdots\cdots$

$c = 5$, and the first five convergents are $\frac{3}{1}, \frac{4}{1}, \frac{7}{2}, \frac{11}{3}, \frac{18}{5}$;

when $n = 2$, $\quad p_n^2 - 13q_n^2 = 4^2 - 13 \times 1^2 = +3$;

$\therefore$ by (18) we have

$$x - y\sqrt{13} = \pm(4 - \sqrt{13})(18 - 5\sqrt{13})^{2m}.$$

Taking $\quad m = 0, \quad -1, \quad +1 \ldots\ldots$

we find $\quad \left.\begin{matrix} x = 4 \\ y = 1 \end{matrix}\right\} \left.\begin{matrix} 256 \\ 71 \end{matrix}\right\} \left.\begin{matrix} 4936 \\ 1369 \end{matrix}\right\} \ldots\ldots$

6. Two lemmas on continued fractions.

Lemma 1. If x lies between $\frac{p_n}{q_n}$ and $\frac{p_{n+1}}{q_{n+1}}$,

where $\frac{p_n}{q_n}$ is the preceding convergent to $\frac{p_{n+1}}{q_{n+1}}$ when converted into a continued fraction, then $\frac{p_n}{q_n}$ is a convergent to the continued fraction which represents x.

For, let $\quad \frac{p_{n+1}}{q_{n+1}} = a_1 + \frac{1}{a_2+} \cdots\cdots\cdots \frac{1}{a_{n+1}}$;

then by supposition x lies between $\frac{p_n}{q_n}$ and $\frac{p_{n+1}}{q_{n+1}}$;

$$\therefore \quad \frac{p_n}{q_n} \sim x < \frac{p_n}{q_n} \sim \frac{p_{n+1}}{q_{n+1}}$$

$$< \frac{p_n}{q_n} \sim \frac{a_{n+1}p_n + p_{n-1}}{a_{n+1}q_n + q_{n-1}}$$

$$< \frac{p_n q_{n-1} \sim p_{n-1} q_n}{q_n(a_{n+1}q_n + q_{n-1})}$$

$$< \frac{1}{q_n(a_{n+1}q_n + q_{n-1})}$$

$$< \frac{1}{q_n(q_n + q_{n-1})}, \qquad \text{since } a_{n+1} \nless 1.$$

$\therefore \ \dfrac{p_n}{q_n}$ is a convergent to the continued fraction which represents x.*

Lemma 2. If $x = \dfrac{\mathrm{M}x' + \mathrm{M}'}{\mathrm{N}x' + \mathrm{N}'}$,

where x' is one of the complete quotients in the development of x as a continued fraction, and M, N, M′, N′ integers all of the same sign, such that $\mathrm{MN}' - \mathrm{M}'\mathrm{N} = \pm 1$

and $\mathrm{M} > \mathrm{M}'$, $\mathrm{N} > \mathrm{N}'$ in absolute magnitude,

then $\dfrac{\mathrm{M}'}{\mathrm{N}'}$ and $\dfrac{\mathrm{M}}{\mathrm{N}}$ are consecutive convergents to the continued fraction which represents x.

We may suppose M, N, M′, N′ to be all positive; for if they were all negative, they could be made all positive by changing the signs in the numerator and the denominator.

Then from the above conditions it can be proved, as in Serret, p. 36, that $\dfrac{\mathrm{M}'}{\mathrm{N}'}$ is the preceding convergent to $\dfrac{\mathrm{M}}{\mathrm{N}}$ when converted into a continued fraction.

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* See *Chrystal's Algebra*, Vol. 2, Chap. XXXII., § 9, Cor. 4.

Now $$\frac{M}{N} \sim x = \frac{M}{N} \sim \frac{Mx' + M'}{Nx' + N'}$$

$$= \frac{MN' \sim M'N}{N(Nx' + N')}$$

$$= \frac{1}{N(Nx' + N')};$$

$$\therefore \quad \frac{M}{N} \sim x < \frac{1}{N(N + N')}, \qquad \text{since } x' > 1.$$

$\therefore \frac{M}{N}$ is a convergent to x, the preceding convergent being $\frac{M'}{N'}$.*

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7. To connect the convergents of the continued fraction representing $\frac{E - \sqrt{C}}{D}$ with those of the continued fraction representing $\frac{E + \sqrt{C}}{D}$.

Let $$\frac{E + \sqrt{C}}{D} = a_1 + \frac{1}{a_2 +} \cdots\cdots \frac{1}{a_k +} \frac{1}{\dot{a}_1 +} \frac{1}{a_2 +} \cdots\cdots \frac{1}{\dot{a}_c +} \cdots\cdots$$

$$= a_1 + \frac{1}{a_2 +} \cdots\cdots \frac{1}{a_k +} \frac{1}{\frac{E' + \sqrt{C}}{D'}};$$

$$\therefore \quad \frac{E + \sqrt{C}}{D} = \frac{\frac{E' + \sqrt{C}}{D'} p_k + p_{k-1}}{\frac{E' + \sqrt{C}}{D'} q_k + q_{k-1}} \quad - \quad - \quad - \quad - \quad - \quad (19)$$

and, changing the sign of $\sqrt{C}$ we have

$$\frac{E - \sqrt{C}}{D} = \frac{\frac{E' - \sqrt{C}}{D'} p_k + p_{k-1}}{\frac{E' - \sqrt{C}}{D'} q_k + q_{k-1}} = \frac{-\frac{E' - \sqrt{C}}{D'} p_k - p_{k-1}}{-\frac{E' - \sqrt{C}}{D'} q_k - q_{k-1}} \quad - \quad (20)$$

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* See *Chrystal's Algebra*, Vol. 2, Chap. XXXII., § 9, Cor. 4.

Now let

$$-\frac{E'-\sqrt{C}}{D'}=\frac{1}{a_{mc}+}\frac{1}{a_{mc-1}+}\cdots\cdots\frac{1}{a_{mc-(r-1)}+}\frac{1}{\dfrac{E''+\sqrt{C}}{D''}}$$

$$=\frac{\dfrac{E''+\sqrt{C}}{D''}P'_r+P'_{r-1}}{\dfrac{E''+\sqrt{C}}{D''}Q'_r+Q'_{r-1}}, \qquad \text{(where } r>1).$$

Substitute in (20) and reduce

$$\therefore\ \frac{E-\sqrt{C}}{D}=\frac{\dfrac{E''+\sqrt{C}}{D''}(p_kP'_r-p_{k-1}Q'_r)+(p_kP'_{r-1}-p_{k-1}Q'_{r-1})}{\dfrac{E''+\sqrt{C}}{D''}(q_kP'_r-q_{k-1}Q'_r)+(q_kP'_{r-1}-q_{k-1}Q'_{r-1})}$$

$$=\frac{\dfrac{E''+\sqrt{C}}{D''}M+M'}{\dfrac{E''+\sqrt{C}}{D''}N+N'} \quad \text{(say)} \quad \cdot\ \cdot\ \cdot\ \cdot \quad (21).$$

$\dfrac{E''+\sqrt{C}}{D''}$ is one of the complete quotients of $-\dfrac{E'-\sqrt{C}}{D'}$

or of $\dfrac{E-\sqrt{C}}{D}$, since $\dfrac{E-\sqrt{C}}{D}$ and $-\dfrac{E'-\sqrt{C}}{D'}$ terminate by the same quotients. (Serret, p. 49.)

The conditions that M, N, M′, N′, may be all of the same sign, are that $\dfrac{p_{k-1}}{p_k}$ and $\dfrac{q_{k-1}}{q_k}$ shall not lie between $\dfrac{P'_{r-1}}{Q'_{r-1}}$ and $\dfrac{P'_r}{Q'_r}$ and these conditions are fulfilled.

For if $\dfrac{p_{k-1}}{p_k}$ and $\dfrac{q_{k-1}}{q_k}$ lay between $\dfrac{P'_{r-1}}{Q'_{r-1}}$ and $\dfrac{P'_r}{Q'_r}$,

then by Lemma 1, $\dfrac{P'_{r-1}}{Q'_{r-1}}$ would be a convergent to $\dfrac{p_{k-1}}{p_k}$ and $\dfrac{q_{k-1}}{q_k}$.

Now $$\frac{p_k}{q_k}=a_1+\frac{1}{a_2+}\cdots\cdots\cdots\frac{1}{a_{k-1}+}\frac{1}{a_k};$$

$$\therefore\ \frac{p_{k-1}}{p_k}=\frac{1}{a_k+}\frac{1}{a_{k-1}+}\cdots\cdots\cdots\frac{1}{a_2+}\frac{1}{a_1}$$

$$\frac{q_{k-1}}{q_k}=\frac{1}{a_k+}\frac{1}{a_{k-1}+}\cdots\cdots\cdots\frac{1}{a_2}$$

and $\dfrac{P'_{r-1}}{Q'_{r-1}} = \dfrac{1}{a_c +} \dfrac{1}{a_{c-1} +} \cdots\cdots\cdots\cdots \dfrac{1}{a_{c-(r-2)}}$;

hence we would have $a_k = a_c,\ a_{k-1} = a_{c-1}, \ldots\ldots a_{k-(r-2)} = a_{c-(r-2)}$.

But $\qquad a_k \neq a_c$;

for, if this were so, the period of $\dfrac{E + \sqrt{C}}{D}$ ought to begin one step earlier.

$\therefore \quad \dfrac{p_{k-1}}{p_k}$ and $\dfrac{q_{k-1}}{q_k}$ do not lie between $\dfrac{P'_{r-1}}{Q'_{r-1}}$ and $\dfrac{P'_r}{Q'_r}$;

$\therefore \quad$ M, N, M′, N′ are all of the same sign.

Again,
$$\begin{aligned} MN' - M'N &= (p_k P'_r - p_{k-1} Q'_r)(q_k P'_{r-1} - q_{k-1} Q'_{r-1}) \\ &\qquad - (p_k P'_{r-1} - p_{k-1} Q'_r)(q_k P'_r - q_{k-1} Q'_r) \\ &= -(p_k q_{k-1} - p_{k-1} q_k)(P'_r Q'_{r-1} - P'_{r-1} Q'_r) \\ &= -(-1)^k(-1)^r \\ &= \pm 1; \end{aligned}$$

thus M is prime to N.

If now $\dfrac{M}{M'} > 1$ and $\dfrac{N}{N'} > 1$, then by Lemma 2

$\dfrac{M'}{N'}$ and $\dfrac{M}{N}$ will be consecutive convergents to $\dfrac{E - \sqrt{C}}{D}$.

But if $\dfrac{M}{M'} < 1$ and $\dfrac{N}{N'} < 1$, continue the development of $-\dfrac{E' - \sqrt{C}}{D'}$ to one more quotient, so that $\dfrac{E'' + \sqrt{C}}{D''} = a_{mc-r} + \cfrac{1}{\cfrac{E''' + \sqrt{C}}{D'''}}$;

substituting in (21) we obtain

$$\frac{E - \sqrt{C}}{D} = \frac{\dfrac{E''' + \sqrt{C}}{D'''}(Ma_{mc-r} + M') + M}{\dfrac{E''' + \sqrt{C}}{D'''}(Na_{mc-r} + N') + N},$$

and now $\dfrac{Ma_{mc-r} + M'}{M} > 1,\ \dfrac{Na_{mc-r} + N'}{N} > 1$;

$\therefore \quad$ as above it follows that $\dfrac{M}{N}$ and $\dfrac{Ma_{mc-r} + M'}{Na_{mc-r} + N'}$

are consecutive convergents to $\dfrac{E - \sqrt{C}}{D}$;

$$\therefore \quad \frac{\text{M}}{\text{N}} \text{ or } \frac{p_k\text{P}'_r - p_{k-1}\text{Q}'_r}{q_k\text{P}'_r - q_{k-1}\text{Q}'_r} \text{ is a convergent to } \frac{\text{E} - \sqrt{\text{C}}}{\text{D}}.$$

The general idea of the above is due to Serret, pp. 70–71,* but the details have been considerably modified.

Let $\frac{p'}{q'}$ denote the convergent $\frac{p_k\text{P}'_r - p_{k-1}\text{Q}'_r}{q_k\text{P}'_r - q_{k-1}\text{Q}'_r}$.

Since the numerator is prime to the denominator,

$$\therefore \quad \left.\begin{aligned} p' &= p_k\text{P}'_r - p_{k-1}\text{Q}'_r \\ q' &= q_k\text{P}'_r - q_{k-1}\text{Q}'_r \end{aligned}\right\} \quad - \quad - \quad - \quad - \quad (22).$$

Then by eliminating p_{k-1} and q_{k-1} from (19) by means of (22) in the same way as in § 1 we deduce that

$$p' - \frac{\text{E} + \sqrt{\text{C}}}{\text{D}}q' = \left(p_k - \frac{\text{E} + \sqrt{\text{C}}}{\text{D}}q_k\right)\left(\text{P}'_r + \frac{\text{E}' + \sqrt{\text{C}}}{\text{D}'}\text{Q}'_r\right).$$

Now let $r = mc - s - 1$, then if $mc - s - 1 > 1$ or $m > \frac{s+2}{c}$,

$$p' - \frac{\text{E} + \sqrt{\text{C}}}{\text{D}}q'$$

$$= \left(p_k - \frac{\text{E} + \sqrt{\text{C}}}{\text{D}}q_k\right)\left(\text{P}'_{mc-s-1} + \frac{\text{E}' + \sqrt{\text{C}}}{\text{D}'}\text{Q}'_{mc-s-1}\right)$$

$$= \left(p_k - \frac{\text{E} + \sqrt{\text{C}}}{\text{D}}q_k\right)\left(\text{P}_s - \frac{\text{E}' + \sqrt{\text{C}}}{\text{D}'}\text{Q}_s\right)\left(\text{P}_c - \frac{\text{E}' + \sqrt{\text{C}}}{\text{D}'}\text{Q}_c\right)^{-m} \text{ by (12)}$$

$$= \left(p_{k+s} - \frac{\text{E} + \sqrt{\text{C}}}{\text{D}}q_{k+s}\right)\left(\text{P}_c - \frac{\text{E}' + \sqrt{\text{C}}}{\text{D}'}\text{Q}_c\right)^{-m} \quad - \quad - \quad - \quad \text{by (5).}$$

Put $k + s = n$ so that $n \nless k$.

$$\therefore \quad p' - \frac{\text{E} + \sqrt{\text{C}}}{\text{D}}q' = \left(p_n - \frac{\text{E} + \sqrt{\text{C}}}{\text{D}}q_n\right)\left(\text{P}_c - \frac{\text{E}' + \sqrt{\text{C}}}{\text{D}'}\text{Q}_c\right)^{-m} \quad - \quad (23)$$

provided $n \nless k$ and $m > \frac{n - k + 2}{c}$.

Comparing (23) with (7) we see that the power of $\text{P}_c - \frac{\text{E}' + \sqrt{\text{C}}}{\text{D}'}\text{Q}_c$ is $-m$ instead of m.

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* See also Legendre's Theorie des Nombres, 3me éd. t. i., §§ 59-74.

Serret's § 29, in which infinite limits are used, and the latter part of § 30 have been dispensed with.

Example: $\dfrac{123+\sqrt{37}}{28}=4+\dfrac{1}{1+}\,\dfrac{1}{1+}\,\dfrac{1}{1+}\,\dfrac{1}{\dot{1}+}\,\dfrac{1}{3+}\,\dfrac{1}{\dot{2}+}\,\cdots\cdots\cdots$

$k=4$, $c=3$, and the first four convergents are $\dfrac{4}{1},\ \dfrac{5}{1},\ \dfrac{9}{2},\ \dfrac{14}{3}$;

and $\dfrac{3+\sqrt{37}}{7}=1+\dfrac{1}{\dot{3}+}\,\dfrac{1}{\dot{2}+}\,\cdots\cdots\cdots$

the first three convergents being $\dfrac{1}{1},\ \dfrac{4}{3},\ \dfrac{9}{7}$.

The requisite conditions are $n \nless 4$ and $m>\dfrac{n-2}{3}$.

Taking $n=4$, $m=1$, these conditions are satisfied.

$\therefore$ by (23)

$$p'-\frac{123+\sqrt{37}}{28}q'=\left(14-\frac{123+\sqrt{37}}{28}\times 3\right)\left(9-\frac{3+\sqrt{37}}{7}\times 7\right)^{-1}$$

$$=\frac{-27-5\sqrt{37}}{28}$$

$$\therefore\ q'=5$$

and $p'=\dfrac{123}{28}\times 5-\dfrac{27}{28}=21.$

Similarly if $\left.\begin{matrix}n=4\\ m=2\end{matrix}\right\}$ then $\dfrac{p'}{q'}=\dfrac{238}{57}$

and if $\left.\begin{matrix}n=4\\ m=3\end{matrix}\right\}$, $\dfrac{p'}{q'}=\dfrac{2877}{689}$.

Thus $\dfrac{21}{5},\ \dfrac{238}{57},\ \dfrac{2877}{689}$ are convergents to

$$\frac{123-\sqrt{37}}{28}=4+\frac{1}{5+}\,\frac{1}{\dot{1}+}\,\frac{1}{2+}\,\frac{1}{\dot{3}+}\,\cdots\cdots\cdots$$

namely the 2nd, 5th, and 8th convergents.

Elementary proof that the mid-points of the three diagonals of any quadrilateral are collinear.

By R. F. Davis.

Seventh Meeting, 12th June 1903.

Professor Gibson in the Chair.

Discussion on the Coordination of the Teaching of Practical and Theoretical Geometry.

APPENDIX.

The Society received an invitation from the Royal University of Christiania to send a representative to the celebration of Abel's Centenary. The invitation arrived too late to be considered by the Society. The invitation and the reply to it are here subjoined.

LE SECRÉTAIRE
de l'Université Royale Frédéric de Norvège

CHRISTIANIA, *Juillet* 1902

MONSIEUR,

En date de 1ier avril nous avons eu l'honneur de remettre à la poste une invitation pour votre institution à se faire représenter à l'occasion des fêtes du Centénaire du mathématicien N. H. Abel le 5–7 Septembre cette année à Christiania.

N'ayant encore reçu votre réponse et craignant que la lettre ne soit venue à votre adresse je me permets de vous demander de bien vouloir nous donner un avis la dessus avant le 1ier Aout.

Veuillez agréer, Monsieur, l'assurance de ma haute considération.

Par l'ordre du Sénat
le secrétaire :
CHR. AUG. ORLAND

PARIS, *le* 29 *Juillet* 1902

MONSIEUR,

Les réunions de la Société Mathématique d'Edimbourg se terminent au mois de Juin, et votre lettre, datée de Juillet 1902,

est arrivée trop tard pour être soumise à une séance de la Société. En conséquence, le Secrétaire m' a remis votre invitation de faire représenter notre Société, et m' a prié, en tout cas, de répondre à votre lettre.

Je regrette profondément qu' aucun de nos membres puisse se rendre à Christiania à l'occasion des fêtes du Centénaire de N. H. Abel, mathématicien dont la carrière si courte a attiré l'admiration du monde scientifique entier.

Je suis convaincu que j'exprime les sentiments de notre Société en vous souhaitant des fêtes aussi brillantes que la carrière d'Abel, et en envoyant à l'Université Royale Frédéric de Norvège nos voeux sincères pour le succès prolongé de son enseignement.

Veuillez agréer, Monsieur, l'assurance de ma haute considération.

John S. Mackay, M.A., LL.D., F.R.S.E.,
Ancien Président de la Société Mathématique d'Edimbourg.

A Monsieur
le Secrétaire de l'Université Royale
Frédéric de Norvège

Edinburgh Mathematical Society.

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18. Tōkyo Sūgaku-Buturigaku Kwai Kizi.
Maki No. IX., Dai 1.

19. Jornal de Sciencias Mathematicas e Astronomicas.
Publicado pelo Dr F. GOMES TEIXEIRA.
Vol. XIV.—No. 6. Vol. XV.—Nos. 1-2.

20. Rendiconti del Circolo Mathematico di Palermo.
Tomo XVI. and XVII.—Nos. 1-3.

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Nieuve Reeks—Achtste Deel—1899-1902.

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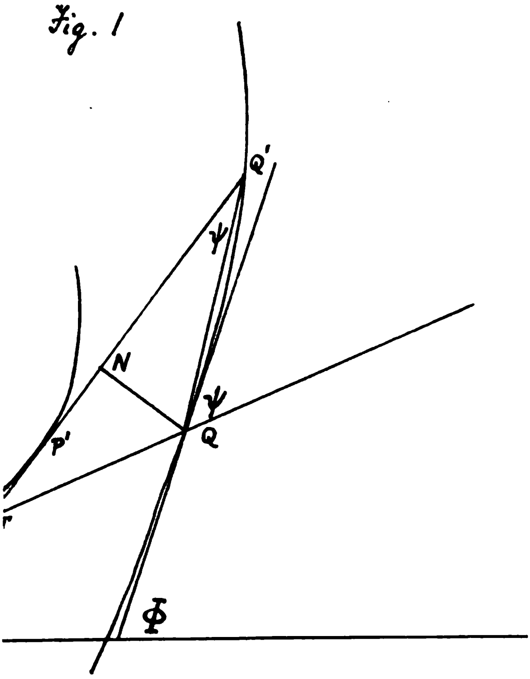
Fig. 1



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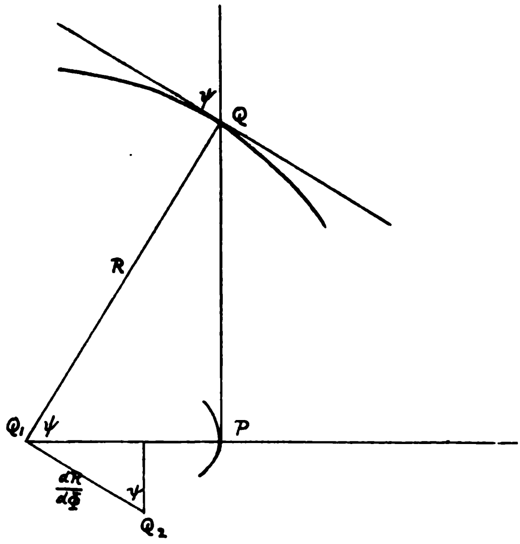

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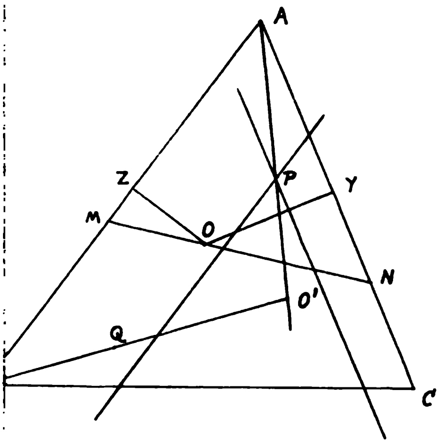

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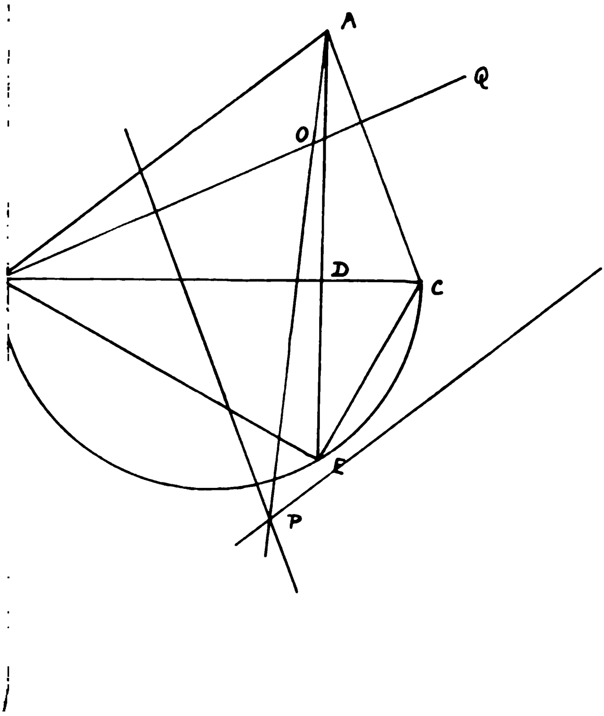

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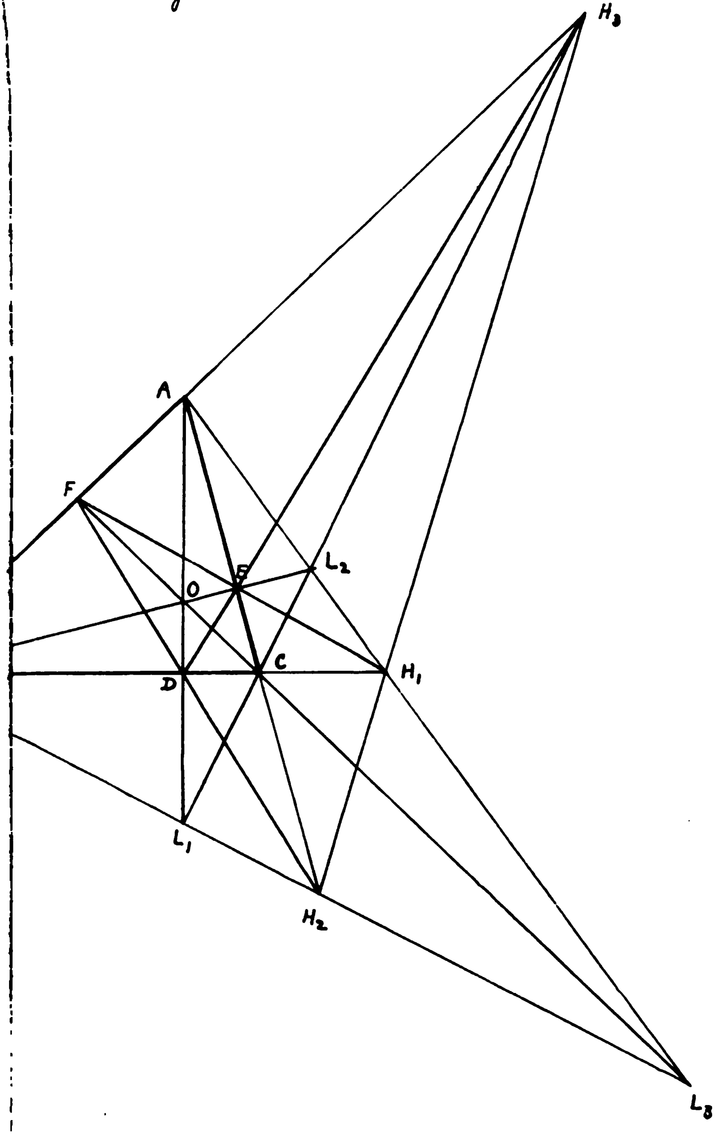


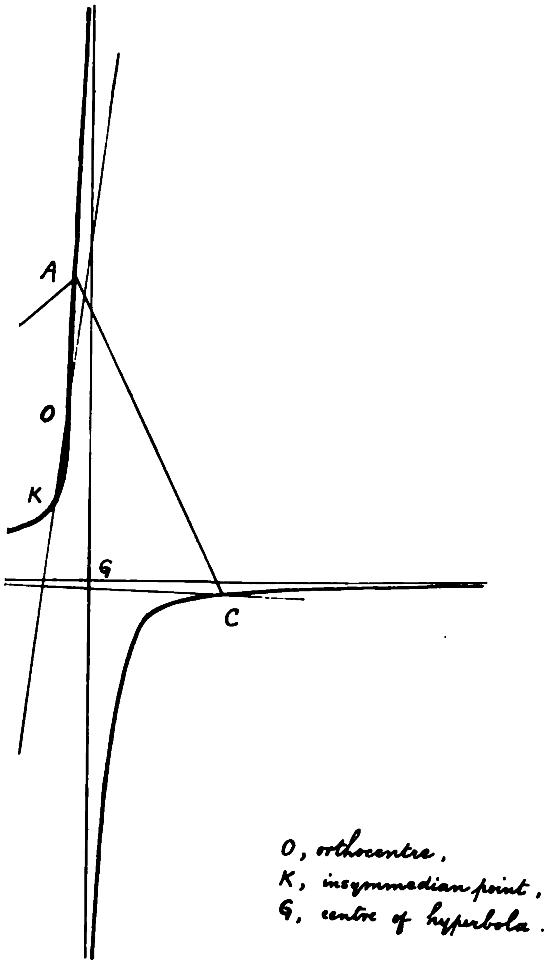
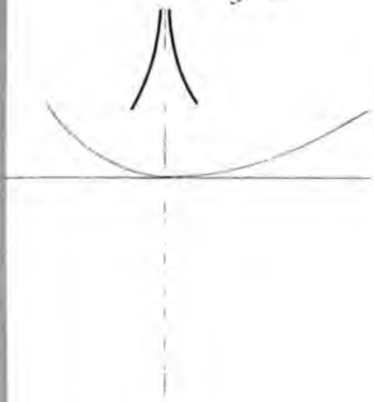

A
O
K
G
C
O, orthocentre,
K, insymmedian point,
G, centre of hyperbola.

Fig. 9.



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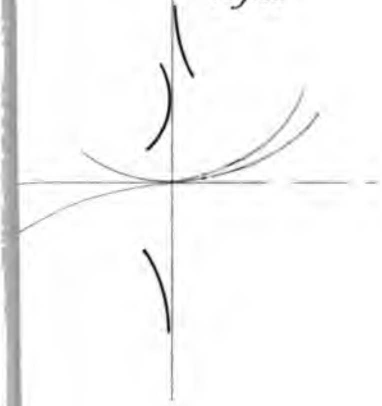

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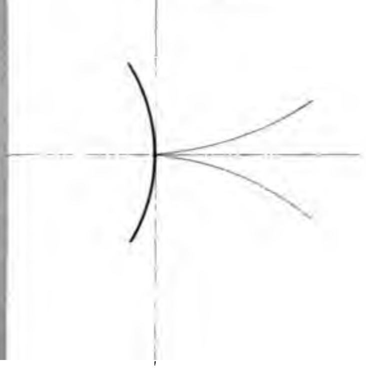

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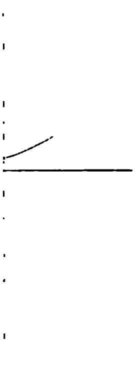

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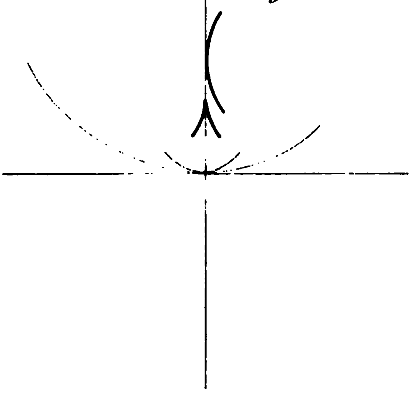

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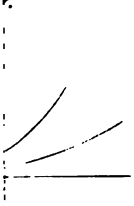

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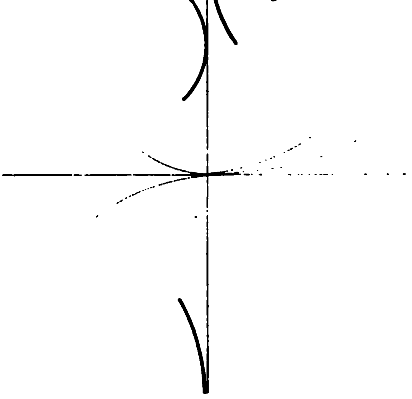

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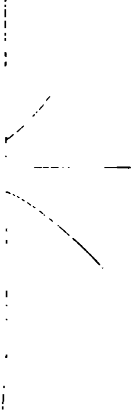

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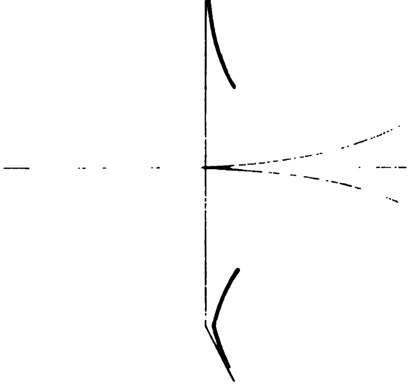

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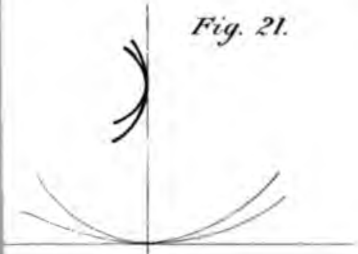

22.

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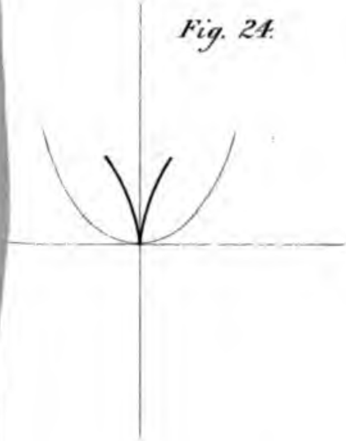

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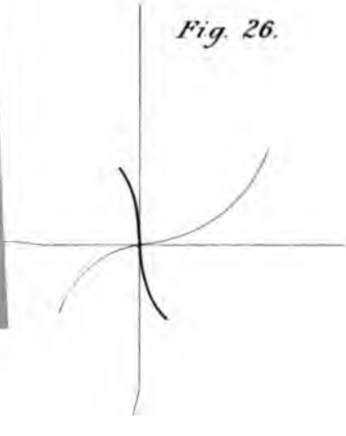

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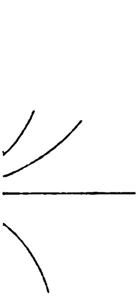

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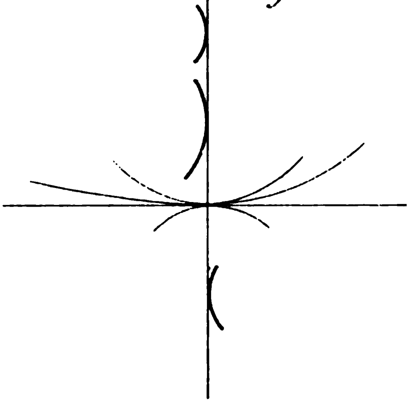
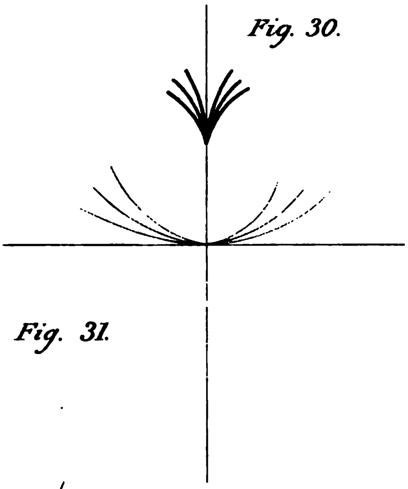


Fig. 29.

Fig. 30.



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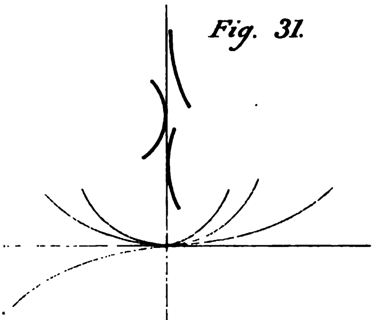

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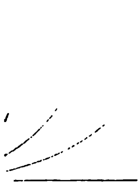

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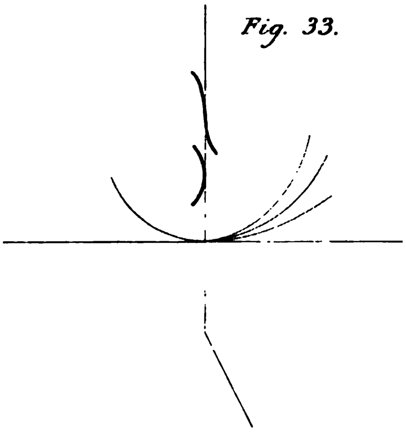

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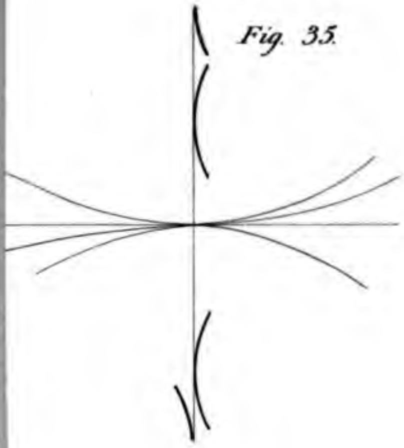

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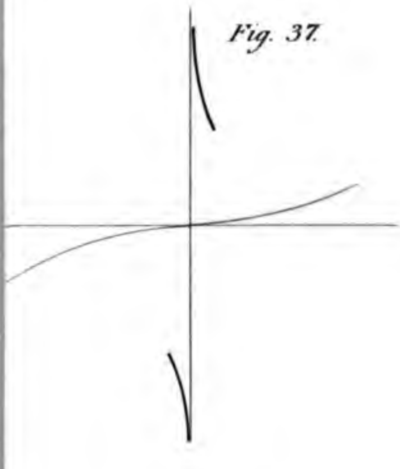
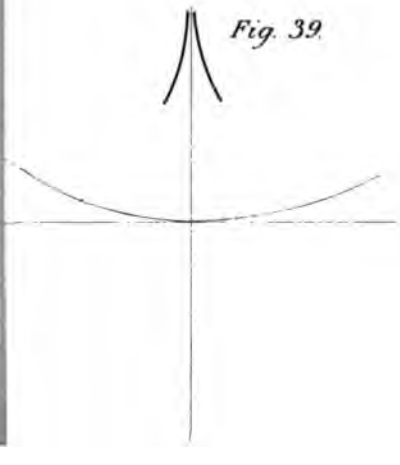


Fig. 39.



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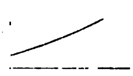
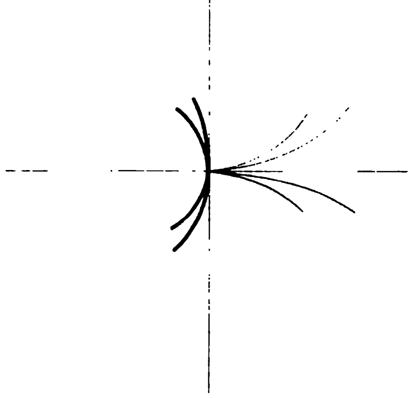


Fig. 41.



7. 42.

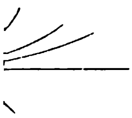
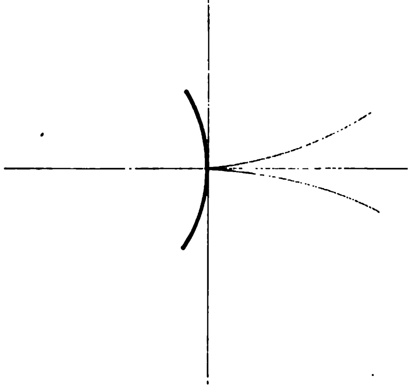


Fig. 43.



7. 44

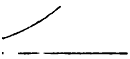
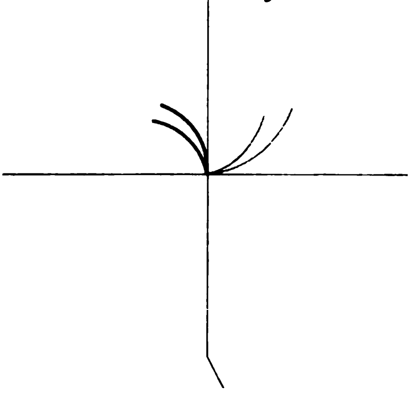


Fig. 45.



40.

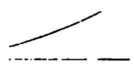
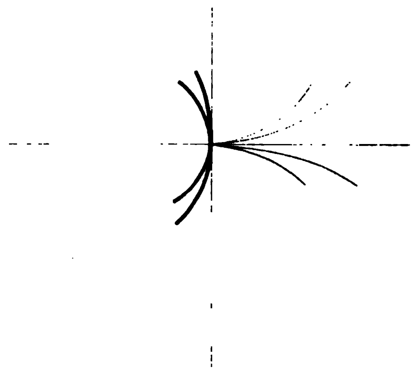


Fig. 41.



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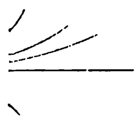
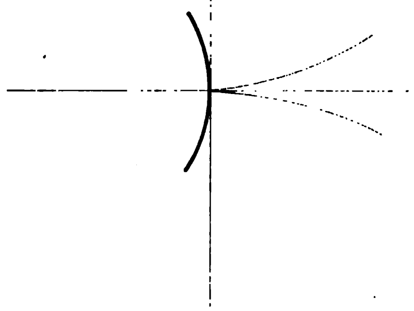


Fig. 43.



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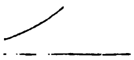


Fig. 45.

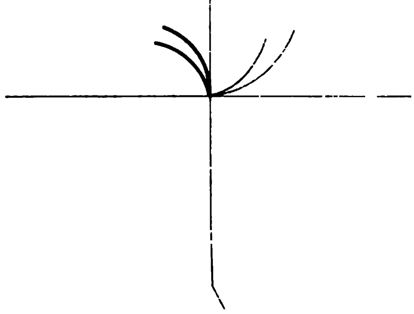
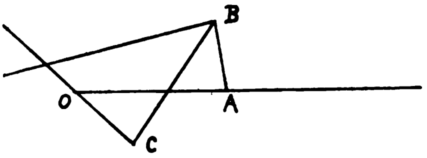


Fig. 48



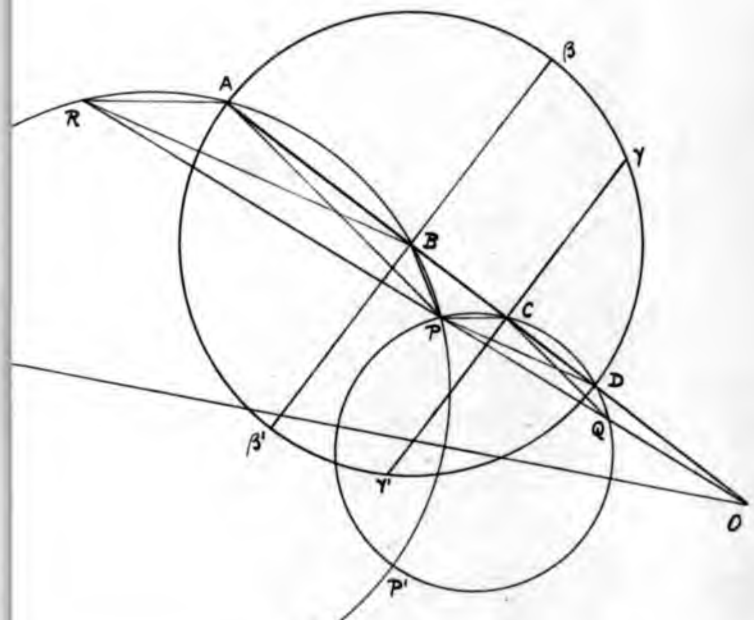


Fig. 49

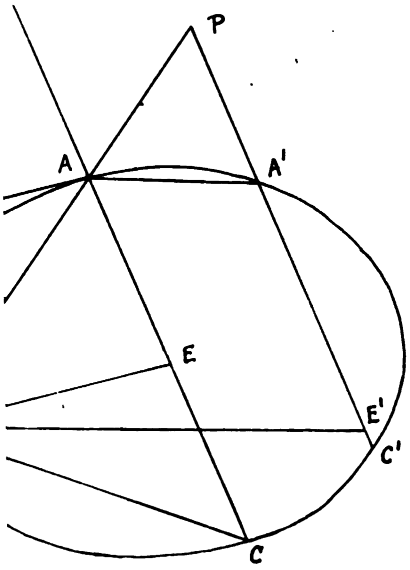


Fig. 50

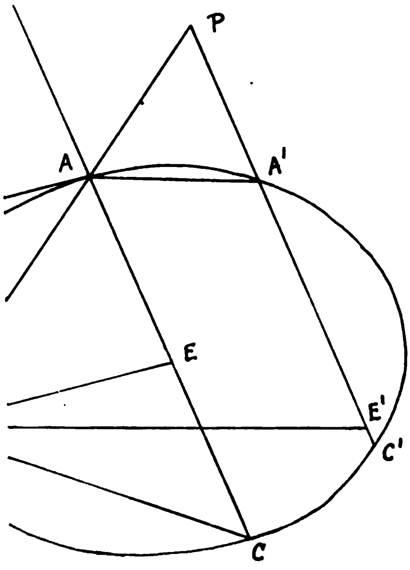


Fig. 50

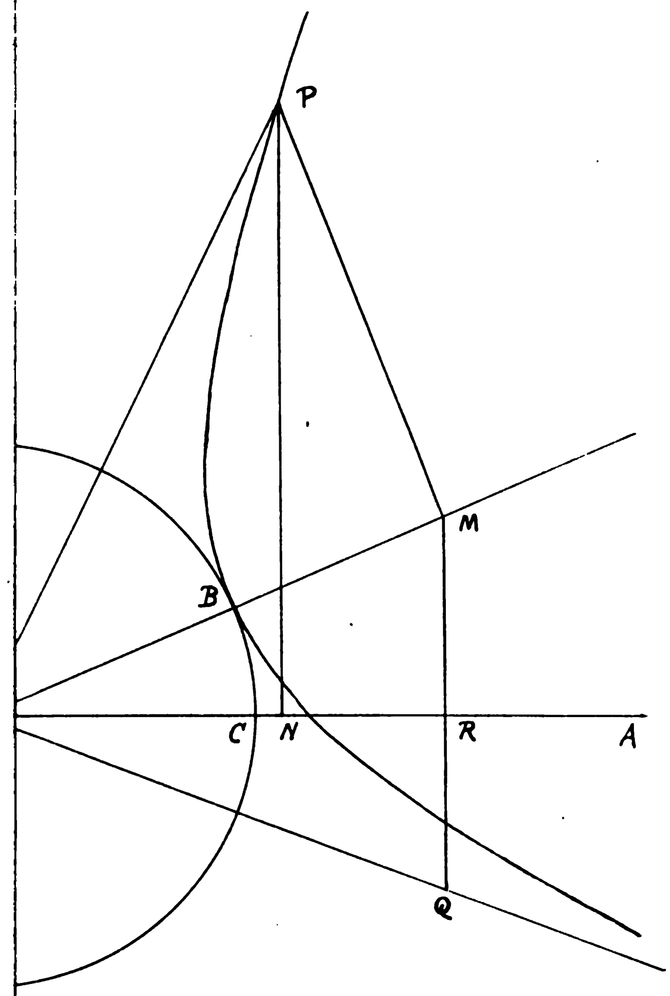


Fig. 51

PROCEEDINGS

OF THE

EDINBURGH MATHEMATICAL SOCIETY.

VOLUME XXII.

SESSION 1903-1904.

WILLIAMS AND NORGATE,
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1904.

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INDEX.

PROCEEDINGS

OF THE

EDINBURGH MATHEMATICAL SOCIETY.

TWENTY-SECOND SESSION, 1903–1904.

First Meeting, 13th November 1903.

Dr THIRD in the Chair.

For this Session the following Office-bearers were elected:—

President - - - Mr CHARLES TWEEDIE, M.A., B.Sc., F.R.S.E.

Vice-President - Mr W. L. THOMSON, M.A., B.A.

Hon. Secretary - DONALD C. M'INTOSH, M.A., F.R.S.E.

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Editors of Proceedings.

Mr W. A. LINDSAY, M.A., B.Sc. (*Convener*).

Mr JOHN TURNER, M.A., B.Sc.

Committee.

Mr JOHN A. THIRD, M.A., D.Sc.

Mr ALEX. G. BURGESS, M.A., F.R.S.E.

Mr J. A. MACDONALD, M.A., B.Sc.

Mr P. PINKERTON, M.A.

Remarques sur l'intégration des fonctions $a^n \cos a\, da$, $a^n \sin a\, da$.

Par M. Edouard Collignon,
Inspecteur général des Ponts et Chaussées en retraite.

Lorsqu' on intègre successivement les fonctions

$$\cos a\, da,\quad a \cos a\, da,\quad a^2 \cos a\, da,\quad a^3 \cos a\, da \ldots\ldots,$$

où le cosinus de l'arc a est multiplié par une puissance à exposant entier de l'arc lui-même,
on reconnaît que les intégrales sont toutes comprises dans la formule

$$P \sin a + Q \cos a,$$

où P et Q représentent des polynomes entiers en a que l'on détermine dans chaque cas particulier.

On a en effet

$$\int \cos a\, da = \sin a,$$

$$\int a \cos a\, da = a \sin a + \cos a,$$

$$\int a^2 \cos a\, da = (a^2 - 2)\sin a + 2a \cos a,$$

$$\ldots\ldots\ldots \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

L'intégration de $\sin a\, da$, $a \sin a\, da$, $a^2 \sin a\, da, \ldots\ldots$ donnerait lieu à des relations analogues.

§ 1

Cherchons d'abord la loi de formation des polynomes P et Q, et posons d'une manière générale

$$(1) \qquad \int a^n \cos a\, da = P_n \sin a + Q_n \cos a,$$

en mettant en évidence l'exposant n dont dépend la forme des polynomes cherchés.

Différentions ; nous devrons avoir identiquement

$$a^n \cos a = \left(P_n + \frac{dQ_n}{da}\right)\cos a + \left(\frac{dP_n}{da} - Q_n\right)\sin a,$$

et par conséquent nous devrons poser

$$(2) \qquad \begin{cases} \dfrac{dP_n}{da} - Q_n = 0, \\[2ex] \dfrac{dQ_n}{da} + P_n = a^n. \end{cases}$$

La première équation montre que Q_n est la dérivée de P_n par rapport à a. Eliminons Q_n entre les deux équations (2). Il viendra

$$(3) \qquad \frac{d^2P_n}{da^2} + P_n = a^n,$$

équation différentielle du second ordre, dont nous devrons prendre seulement la solution particulière dans laquelle P_n est un polynome entier en a. Posons, en appelant A_1, A_2, A_3, ... des coefficients indéterminés,

$$P_n = a^n + A_1a^{n-1} + A_2a^{n-2} + \ldots + A_{n-2}a^2 + A_{n-1}a + A_n.$$

La seconde dérivée sera égale à

$$\frac{d^2P_n}{da^2} = n(n-1)a^{n-2} + (n-1)(n-2)A_1a^{n-3} + (n-2)(n-3)A_2a^{n-4} + \ldots\ldots\ldots + 2A_{n-2},$$

et la somme des deux fonctions entières doit se réduire à a^n ;

$$a^n = a^n + A_1 \left| \begin{array}{r} a^{n-1} + A_2 \\ + n(n-1) \end{array} \right| \begin{array}{r} a^{n-2} + A_3 \\ + (n-1)(n-2)A_1 \end{array} \left| \, a^{n-3} \right.$$

$$+ \ldots\ldots\ldots + \left. \begin{array}{r} A_n \\ + 2A_{n-2} \end{array} \right|$$

d'où l'on déduit

$$\begin{array}{lcl} A_1 = 0 & \text{et par suite} & A_1 = 0, \\ A_2 + n(n-1) = 0 & ,, & A_2 = -n(n-1), \\ A_3 + (n-1)(n-2)A_1 = 0 & ,, & A_3 = 0, \\ A_4 + (n-2)(n-3)A_2 = 0 & ,, & A_4 = n(n-1)(n-2)(n-3), \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots & & A_5 = 0, \\ A_n + 2A_{n-2} = 0. & & \end{array}$$

Les coefficients de rang pair A_1, A_3, A_5... sont tous nuls, et l'on a par conséquent

$$A_{n-1} = 0 \text{ si } n \text{ est pair,} \quad A_n = 0 \text{ si } n \text{ est impair.}$$

Les signes des coefficients A_2, A_4, A_6, ... sont alternativement $-$ et $+$, de sorte qu'on a

$$A_n \ = (-1)^m \times (n!), \quad A_{n-1} = 0, \quad \text{si } n = 2m\,;$$

$$A_{n-1} = (-1)^m \times (n!), \quad A_n \ = 0, \quad \text{si } n = 2m+1.$$

Soit par exemple $n = 6$. On aura

$$A_2 = -6.5 = -30, \quad A_1 = A_3 = A_5 = 0,$$

$$A_4 = +6.5.4.3 = 360,$$

$$A_6 = -6.5.4.3.2.1 = 720\,;$$

et pour $n = 7$

$$A_2 = -7.6 = -42, \quad A_1 = A_3 = A_5 = A_7 = 0,$$

$$A_4 = 840, \quad A_6 = -5040.$$

Cette règle suffit à la rigueur pour écrire immédiatement les facteurs P_n et Q_n de $\sin a$ et de $\cos a$. Mais on peut encore la simplifier, grâce aux remarques suivantes.

Nous venons de trouver

$$(4) \quad P_n = a^n - n(n-1)a^{n-2} + n(n-1)(n-2)(n-3)a^{n-4} - \ldots\ldots$$

et en prenant la première dérivée du polynome on en déduit

$$(5) \quad Q_n = \frac{dP_n}{da} = na^{n-1} - n(n-1)(n-2)a^{n-3} + n(n-1)(n-2)(n-3)(n-4)a^{n-5}$$
$$- \ldots\ldots\ldots\ldots -$$

En comparant ces deux polynomes terme par terme, on reconnaît que na^{n-1} est la dérivée de a^n ; de sorte que le premier terme de Q_n s'obtient par dérivation du premier terme de P_n ;

que $n(n-1)a^{n-2}$, second terme de P_n changé de signe, est la dérivée de na^{n-1}, premier terme de Q_n ;

que $n(n-1)(n-2)a^{n-3}$ est la dérivée de $n(n-1)a^{n-2}$, de sorte que la dérivation du second terme de P_n donne le second terme de Q_n,

et ainsi de suite alternativement, en ayant soin d'alterner les signes des termes obtenus, de manière à prendre négativement dans chaque développement les termes de rang pair, positivement les termes de rang impair.

Pour opérer les développements des deux polynomes, il convient d'écrire alternativement les dérivées formant chaque terme suivant deux lignes horizontales, l'une qui donnera le développement de P_n, l'autre le développement de Q_n :

$$a^n \qquad n(n-1)a^{n-2} \qquad n(n-1)(n-2)(n-3)a^{n-4}$$

$$na^{n-1} \qquad n(n-1)(n-2)a^{n-3} \qquad n(n-1)(n-2)(n-3)(n-4)a^{n-5}$$

et ainsi de suite ; en mettant le signe $-$ aux termes de rang pair dans chaque ligne, il vient en définitive

$$P_n = a^n - n(n-1)a^{n-2} + n(n-1)(n-2)(n-3)a^{n-4} - \ldots\ldots$$

$$Q_n = na^{n-1} - n(n-1)(n-2)a^{n-3} + n(n-1)(n-2)(n-3)(n-4)a^{n-5} - \ldots$$

Posons par exemple $n = 7$. Il viendra

$$P_7 = a^7 - 42a^5 + 840a^3 - 5040a,$$

$$Q_7 = 7a^6 - 210a^4 + 2520a^2 - 5040,$$

et l'on aura par conséquent

$$\int a^7 \cos a\, da = (a^7 - 42a^5 + 840a^3 - 5040a)\sin a + (7a^6 - 210a^4 + 2520a^2 - 5040)\cos a.$$

Si, au lieu de mettre en facteur les polynomes P_n, Q_n qui multiplient respectivement $\sin a$ et $\cos a$, on ordonne le second membre par rapport aux puissances descendantes de a, on écrira

$$\int a^7 \cos a\, da = a^7 \sin a + 7a^6 \cos a - 42a^5 \sin a - 210a^4 \cos a + 840a^3 \sin a + 2520a^2 \cos a - 5040a \sin a - 5040 \cos a,$$

et il est clair que pareille disposition est applicable au cas général :

$$(6) \qquad \int a^n \cos a\, da = a^n \sin a + na^{n-1} \cos a - n(n-1)a^{n-2} \sin a - n(n-1)(n-2)a^{n-3} \cos a + \ldots\ldots$$

Or pour passer d'un terme au suivant, il suffit de prendre séparément les dérivées des deux facteurs qui composent le terme considéré, savoir le monôme contenant la puissance de a et le facteur trigonométrique qui la multiplie ; par exemple, si l'on isole les facteurs

$$a^n \text{ et } \sin a$$

qui forment le premier terme, on obtiendra le second en prenant leurs dérivées respectives

$$na^{n-1} \text{ et } \cos a,$$

ce qui donnera $na^{n-1}\cos\alpha$;

pour passer au suivant, on isolera de même

$$na^{n-1} \text{ et } \cos\alpha$$

et prenant les dérivées de chacun des facteurs on aura

$$n(n-1)\alpha^{n-2} \text{ et } -\sin\alpha$$

dont le produit est

$$-n(n-1)\alpha^{n-2}\sin\alpha,$$

c'est à dire, le troisième terme.

Le quatrième sera de même

$$n(n-1)(n-2)\alpha^{n-3} \times (-\cos\alpha)$$
$$= -n(n-1)(n-2)\alpha^{n-3}\cos\alpha,$$

et ainsi de suite, jusqu' à épuisement des dérivées des puissances de α ; les facteurs trigonométriques règlent le signe de chaque terme, par l'opération même de la dérivation.

Exemple. Écrire l'intégrale $\int \alpha^{10}\cos\alpha \, d\alpha$.

On pourra disposer les calculs en trois colonnes verticales, l'une renfermant le monôme α^{10} et ses dérivées successives, l'autre $\sin\alpha$ et ses dérivées, la troisième le produit des dérivées correspondantes prises dans les deux colonnes.

α^{10}	$\sin\alpha$	$\int \alpha^{10}\cos\alpha \, d\alpha = \alpha^{10}\sin\alpha$
$10\alpha^9$	$\cos\alpha$	$+ 10\alpha^9 \cos\alpha$
$90\alpha^8$	$-\sin\alpha$	$- 90\alpha^8 \sin\alpha$
$720\alpha^7$	$-\cos\alpha$	$- 720\alpha^7 \cos\alpha$
$5040\alpha^6$	$\sin\alpha$	$+ 5040\alpha^6 \sin\alpha$
$30240\alpha^5$	$\cos\alpha$	$+ 30240\alpha^5 \cos\alpha$
$151200\alpha^4$	$-\sin\alpha$	$- 151200\alpha^4 \sin\alpha$
$604800\alpha^3$	$-\cos\alpha$	$- 604800\alpha^3 \cos\alpha$
$1814400\alpha^2$	$\sin\alpha$	$+ 1814400\alpha^2 \sin\alpha$
3628800α	$\cos\alpha$	$+ 3628800\alpha \cos\alpha$
3628800	$-\sin\alpha$	$- 3628800\sin\alpha$

On aurait du même coup P_{10} et Q_{10} en ordonnant l'intégrale obtenue par rapport à $\sin a$ et à $\cos a$, et en prenant pour P_{10} le coefficient de $\sin a$, pour Q_{10} le coefficient de $\cos a$, ce qui revient à prendre pour P_{10} les termes inscrits dans la première colonne à gauche, de deux en deux en alternant les signes, et pour Q_{10} les autres termes avec signes alternatifs :

$$P_{10} = a^{10} - 90a^8 + 5040a^6 - 151200a^4 + 1814400a^2 - 3628800,$$
$$Q_{10} = 10a^9 - 720a^7 + 30240a^5 - 604800a^3 + 3628800a.$$

§ 2

On trouverait de même l'intégrale $\int a^n \sin a \, da$. Posons, par analogie avec ce que nous avons fait pour l'intégrale $\int a^n \cos a \, da$,

$$\int a^n \sin a \, da = M \cos a + N \sin a,$$

M et N désignant des polynomes entiers en a, qu'il s'agit de déterminer.

Nous aurons, en différentiant et en divisant par da,

$$\begin{aligned} a^n \sin a &= -M \sin a + N \cos a + \frac{dM}{da} \cos a + \frac{dN}{da} \sin a \\ &= \left(\frac{dN}{da} - M\right) \sin a + \left(\frac{dM}{da} + N\right) \cos a, \end{aligned}$$

d'où résultent les relations qui assurent l'identité des deux membres :

$$(9) \quad \begin{cases} \dfrac{dN}{da} - M = a^n, \\ \dfrac{dM}{da} + N = 0. \end{cases}$$

Comparons le système d'équations (9) au système (2) :

$$(2) \quad \begin{cases} \dfrac{dP_n}{da} - Q_n = 0, \\ \dfrac{dQ_n}{da} + P_n = a^n. \end{cases}$$

On ramène le premier système au second en posant

$$N = Q_n \text{ et } M = -P_n,$$

car la première du groupe (9) devient identique à la seconde du groupe (2), et la seconde du groupe (9) reproduit la première du groupe (2) changée de signe. On aura donc, sans nouveaux calculs,

$$(10) \quad \int a^n \sin a \, da = -P_n \cos a + Q_n \sin a.$$

Si l'on rapproche cette formule de notre formule (1)

$$(1) \quad \int a^n \cos a \, da = P_n \sin a + Q_n \cos a,$$

on pourra les fondre en une seule de deux manières, soit en posant

$$(11) \quad \int a^n \left| \begin{matrix} \cos a \\ \sin a \end{matrix} \right| da = P_n \left| \begin{matrix} \sin a \\ -\cos a \end{matrix} \right| + Q_n \left| \begin{matrix} \cos a \\ \sin a \end{matrix} \right|$$

ce qui revient à changer $\cos a$ en $\sin a$ et $\sin a$ en $-\cos a$ pour passer de l'équation (1) à l'équation (10), ou en multipliant l'équation (10) par l'unité imaginaire $i = \sqrt{-1}$, et en l'ajoutant ensuite à l'équation (1); il vient en effet

$$(12) \quad \int a^n (\cos a + i \sin a) \, da = \int a^n e^{ai} \, da$$
$$= P_n(\sin a - i \cos a) + Q_n(\cos a + i \sin a)$$
$$= P_n \frac{e^{ai}}{i} + Q_n e^{ai} = e^{ai}(Q_n - i P_n).$$

§ 3

Nous avons obtenu les relations générales

$$(4) \quad P_n = a^n - n(n-1)a^{n-2} + n(n-1)(n-2)(n-3)a^{n-4} \ldots\ldots,$$

$$(5) \quad Q_n = na^{n-1} - n(n-1)(n-2)a^{n-3} + n(n-1)(n-2)(n-3)(n-4)a^{n-5} \ldots.$$

Changeons n en $n-1$ dans la première équation ; il viendra

$$(13) \quad P_{n-1} = a^{n-1} - (n-1)(n-2)a^{n-3} + (n-1)(n-2)(n-3)(n-4)a^{n-5} \ldots,$$

et cette équation multipliée par n reproduit la valeur de Q_n. On a donc

(14) $Q_n = nP_{n-1}$,

relation générale qui donne, en y changeant n en $n-1$,

$$Q_{n-1} = (n-1)P_{n-2}.$$

On a d'ailleurs

$$\frac{dP_{n-1}}{da} = Q_{n-1} = (n-1)P_{n-2},$$

et multipliant par n, il vient

$$n\frac{dP_{n-1}}{da} = n(n-1)P_{n-2}.$$

Formons $\frac{dP_{n-1}}{da}$; nous aurons en multipliant par n la dérivée de l'équation (4)

$$n\frac{dP_{n-1}}{da} = n(n-1)a^{n-2} - n(n-1)(n-2)(n-3)a^{n-4}$$
$$+ n(n-1)(n-2)(n-3)(n-4)(n-5)a^{n-6}\ldots\ldots,$$

ce qui reproduit, changés de signes, les termes de P_n à partir du second. On a donc l'équation

(15) $P_n = a^n - n\frac{dP_{n-1}}{da} = a^n - n(n-1)P_{n-2}$

qui établit une relation de recurrence reliant entre elles les fonctions P_n de deux en deux.

On prouverait de même, et on établirait du reste, soit en dérivant l'équation (15), soit en nous servant de l'équation (14),

(16) $Q_n = na^{n-1} - n(n-1)Q_{n-2}$.

Les relations (15) et (16) permettent de former de proche en proche les fonctions P et les fonctions Q. On partira de l'intégrale connue $\int \cos a \, da = \sin a$ qui montre que $P_0 = 1$ et $Q_0 = 0$; puis de

l'intégrale $\int a\cos a\, da = a\sin a + \cos a$, ce qui entraîne $P_1 = a$, $Q_1 = 1$, et l'on formera les deux suites :

$$
\begin{aligned}
P_0 &= 1,\\
P_2 &= a^2 - 2,\\
P_4 &= a^4 - 12(a^2 - 2) = a^4 - 12a^2 + 24,\\
P_6 &= a^6 - 30(a^4 - 12a^2 + 24)\\
&= a^6 - 30a^4 + 360a^2 - 720,\\
P_8 &= a^8 - 56(a^6 - 30a^4 + 360a^2 - 720)\\
&= a^8 - 56a^6 + 1680a^4 - 20160a^2 + 40320,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
P_1 &= a,\\
P_3 &= a^3 - 6a,\\
P_5 &= a^5 - 20(a^3 - 6a) = a^5 - 20a^3 + 120a,\\
P_7 &= a^7 - 42(a^5 - 20a^3 + 120a)\\
&= a^7 - 42a^5 + 840a^3 - 5040a,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}
$$

La série des valeurs de Q_n s'obtiendrait de même en partant des relations

$$Q_0 = 0 \text{ et } Q_1 = +1$$

et du premier terme de la formule, toujours égal à na^{n-1}.

Il vient

$$
\begin{aligned}
Q_2 &= 2a,\\
Q_4 &= 4a^3 - 12 \times 2a = 4a^3 - 24a,\\
Q_6 &= 6a^5 - 30(4a^3 - 24a) = 6a^5 - 120a^3 + 720a,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
Q_3 &= 3a^2 - 6,\\
Q_5 &= 5a^4 - 20(3a^2 - 6) = 5a^4 - 60a^2 + 120, \text{ etc.}
\end{aligned}
$$

Note on M. Collignon's Paper on the Integration of $a^n \cos a\, da$ and $a^n \sin a\, da$.

By Professor Gibson.

The properties of the functions P_n, Q_n may also be determined very briefly as follows :—

Let u be a function of x; then by integration by parts we have

$$\int e^{\frac{x}{a}} u dx = ae^{\frac{x}{a}} u - a\int e^{\frac{x}{a}} \frac{du}{dx} dx$$

$$= ae^{\frac{x}{a}}\left\{u - a\frac{du}{dx} + a^2\frac{d^2u}{dx^2} - \ldots + (-1)^r a^r \frac{d^ru}{dx^r} + \ldots\right\}.$$

Now let $u = x^n$ and let ${}_nP_r = n(n-1)(n-2)\ldots(n-r+1)$, where n is a positive integer. Then

$$\int e^{\frac{x}{a}} x^n dx = ae^{\frac{x}{a}}\left\{x^n - {}_nP_1 a x^{n-1} + {}_nP_2 a^2 x^{n-2} - \ldots + (-1)^r {}_nP_r a^r x^{n-r} + \ldots\right\}.$$

Put $a = i$ so that $e^{\frac{x}{a}} = e^{-xi} = \cos x - i \sin x$; then

$$\int (\cos x - i \sin x) x^n\, dx$$

$$= (\sin x + i \cos x)\{x^n - i\,{}_nP_1 x^{n-1} - {}_nP_2 x^{n-2} + i\,{}_nP_3 x^{n-3} + {}_nP_4 x^{n-4} - \ldots$$
$$+ (-1)^r {}_nP_{2r} x^{n-2r} + i(-1)^{r+1} {}_nP_{2r+1} x^{n-2r-1} + \ldots\}.$$

Equating real and imaginary parts we get

$$\int \cos x\, x^n dx = \sin x\{x^n - {}_nP_2 x^{n-2} + {}_nP_4 x^{n-4} - \ldots + (-1)^r {}_nP_{2r} x^{n-2r} + \ldots\}$$
$$+ \cos x\{{}_nP_1 x^{n-1} - {}_nP_3 x^{n-3} + {}_nP_5 x^{n-5} - \ldots + (-1)^r {}_nP_{2r+1} x^{n-2r-1} + \ldots\}$$
$$= \quad P_n \sin x + Q_n \cos x,$$

$$\int \sin x . x^n dx = -P_n \cos x + Q_n \sin x.$$

The method of deriving the series proves the relation

$$Q_n = \frac{dP_n}{dx};$$

the actual values verify the relation and give also the general term.

Note on a Theorem in Double Series.

By W. L. Thomson, M.A.

Second Meeting, 11th December 1903.

Mr W. L. Thomson, Vice-President, in the Chair.

On some Algebraic Identities.

By F. H. Jackson, M.A.

On a form of Maclaurin's Theorem.

By F. H. Jackson, M.A.

On the quadrilateral circuminscribed to two circles.

By R. F. Davis, M.A.

Figure 1.

Let ABCD be a quadrilateral inscribed in a circle (centre O, radius ρ) whose diagonals AC, BD intersect at right angles in S. From S draw SE, SF, SG, SH perpendiculars on AB, BC, CD, DA respectively.

Then EFGH is a quadrilateral circuminscribed to two circles. It is, moreover, the earliest and simplest form in which such a figure would ordinarily present itself to a student in Geometry.

[From the various cyclic quadrilaterals $S\hat{E}F = S\hat{B}C = S\hat{A}D = S\hat{E}H$ and SE bisects $F\hat{E}H$. Again $F\hat{E}H = 2S\hat{A}D$ and $F\hat{G}H = 2S\hat{D}A$, so that $F\hat{E}H$ and $F\hat{G}H$ are supplementary].

S is the incentre of EFGH. Since $SE^2 = AE \,.\, BE = \rho^2 - OE^2$, therefore E (and similarly F, G, H) lies on the circular locus $SP^2 + OP^2 = \rho^2$ whose centre X is at the middle point of OS and whose radius R is given by the relation

$$2R^2 + 2d^2 = \rho^2 \qquad \text{(i)}$$

where $d = SX$.

Again, if r be the radius of the circle inscribed in EFGH,

$$r = SE\sin SAD = SA\sin SAB \,.\, \sin SAD = SA \,.\, BC \,.\, CD/4\rho^2 = SA \,.\, SC/2\rho,$$

$$\text{thus } 2\rho r = \rho^2 - 4d^2 \qquad \text{(ii).}$$

If we eliminate ρ between (i) and (ii) we get

$$(R+d)^{-2} + (R-d)^{-2} = r^{-2}$$

which is the known poristic relation.

Third Meeting, 8th January 1904.

Mr CHARLES TWEEDIE, President, in the Chair.

On the Fractional Infinite Series for cosec$x$, sec$x$, cot$x$, and tan$x$.

By D. K. PICKEN, M.A.

The infinite products for sin$x$ and cos$x$ are most conveniently obtained in a rigorous way from the well-known factorial expressions for sin$n\theta$ and cos$n\theta$ which, when n is an even integer, take the forms

$$\text{(i) } \sin n\theta = 2^{n-1}.\sin\theta\cos\theta\left(\sin^2\frac{\pi}{n} - \sin^2\theta\right)\left(\sin^2\frac{2\pi}{n} - \sin^2\theta\right)\ldots\left(\sin^2\frac{\overline{n-2}.\pi}{2n} - \sin^2\theta\right)$$

$$\text{(ii) } \cos n\theta = 2^{n-1}.\left(\sin^2\frac{\pi}{2n} - \sin^2\theta\right)\left(\sin^2\frac{3\pi}{2n} - \sin^2\theta\right)\ldots\left(\sin^2\frac{\overline{n-1}.\pi}{2n} - \sin^2\theta\right);$$

θ being put equal to $\dfrac{x}{n}$ and n made infinitely great.*

It is then usual to obtain the fractional infinite series for cot$x$ and tan$x$ by logarithmic differentiation—a process in which the treatment of the remainder is somewhat involved—and to deduce those for cosec$x$ and sec$x$ by the use of certain elementary trigonometrical identities.

It seems, however, a more fundamental process to obtain from (i) and (ii) expressions for cosθcosec$n\theta$, sec$n\theta$, secθcot$n\theta$, and secθtan$n\theta$ in partial fractions; the degree in sinθ of the denominator in each of these functions being higher than that of the numerator; and then to proceed to the limit as in the case of the products.

* Cf. *Hobson's Trigonometry*, Chap. XVII.

I. When n is even

$$\frac{\cos\theta}{\sin\theta\sin n\theta} \text{ can be written in the form } \sum_{0}^{\frac{n}{2}-1} \frac{A_r}{\sin^2\frac{r\pi}{n} - \sin^2\theta},$$

$$\text{where}\quad A_0 = \left[-\frac{\sin\theta\cos\theta}{\sin n\theta}\right]_{\theta=0} = -\frac{1}{n}$$

$$\text{and}\quad A_r = \left[\frac{\left(\sin^2\frac{r\pi}{n} - \sin^2\theta\right)\cos\theta}{\sin\theta\,.\,\sin n\theta}\right]_{\theta=\frac{r\pi}{n}}$$

$$= \left[\frac{\left(\sin\frac{r\pi}{n} + \sin\theta\right)\cos\theta}{\sin\theta}\,.\,\frac{\sin\frac{r\pi}{n} - \sin\theta}{\sin n\theta}\right]_{\theta=\frac{r\pi}{n}}$$

$$= \frac{-2\cos^2\frac{r\pi}{n}}{n\cos r\pi} = \frac{(-)^{r-1}}{n}\,.\,2\cos^2\frac{r\pi}{n},\ \left(r = 1,\ 2,\ \ldots\ \frac{n}{2} - 1\right);$$

$$\therefore\ \cos\theta\,\mathrm{cosec}\,n\theta = \frac{1}{n\sin\theta} + \frac{2\sin\theta}{n}\sum_{1}^{\frac{n}{2}-1}(-)^{r-1}\frac{\cos^2\frac{r\pi}{n}}{\sin^2\frac{r\pi}{n} - \sin^2\theta}.$$

[When n is odd,

$$\mathrm{cosec}\,n\theta = \frac{1}{n\sin\theta} + \frac{2\sin\theta}{n}\sum_{1}^{\frac{n-1}{2}}(-)^{r-1}.\frac{\cos\frac{r\pi}{n}}{\sin^2\frac{r\pi}{n} - \sin^2\theta}\ ].$$

Putting $\theta = \frac{x}{n}$, we get

$$\cos\frac{x}{n}\,\mathrm{cosec}\,x = \frac{1}{n\sin\frac{x}{n}} + \frac{2}{n}\sin\frac{x}{n}\sum_{1}^{k}(-)^{r-1}.\frac{\cos^2\frac{r\pi}{n}}{\sin^2\frac{r\pi}{n} - \sin^2\frac{x}{n}} + (-)^k.\,\mathrm{R}$$

where k is any integer less than $\left(\frac{n}{2} - 1\right)$.

It is obvious that R is positive and less than

$$\frac{2\sin\frac{x}{n}\cos^2\frac{(k+1)\pi}{n}}{n\left\{\sin^2\frac{(k+1)\pi}{n}-\sin^2\frac{x}{n}\right\}}$$

provided n is so great that k can be chosen greater than $\frac{x}{\pi}$; for the angles $\frac{r\pi}{n}$ are increasing *acute* angles and therefore the terms of R are in descending order of numerical magnitude.

$$\therefore\ \cos\frac{x}{n}\operatorname{cosec}x=\frac{1}{n\sin\frac{x}{n}}+2n\sin\frac{x}{n}\,.\,\sum_1^k(-)^{r-1}\,.\,\frac{\cos^2\frac{r\pi}{n}}{n^2\left\{\sin^2\frac{r\pi}{n}-\sin^2\frac{x}{n}\right\}}$$

$$+(-)^k\,.\,\epsilon\,\frac{2n\sin\frac{x}{n}\cos^2\frac{(k+1)\pi}{n}}{n^2\left\{\sin^2\frac{(k+1)\pi}{n}-\sin^2\frac{x}{n}\right\}},$$

where ϵ is a positive proper fraction;

$\therefore$ proceeding to the limit when n becomes infinitely great,

$$\operatorname{cosec}x=\frac{1}{x}+\sum_1^k(-)^{r-1}\,.\,\frac{2x}{r^2\pi^2-x^2}+(-)^k\,.\,\epsilon_1\,.\,\frac{2x}{(k+1)^2\pi^2-x^2},$$

ϵ_1 being the limiting positive fractional value of ϵ.

Hence the greater we make the finite number k the more nearly is $\operatorname{cosec}x$ equal to $\frac{1}{x}+\sum_1^k(-)^{r-1}\,.\,\frac{2x}{r^2\pi^2-x^2}$, and the difference vanishes when k becomes infinitely great.

i.e., $\operatorname{cosec}x=\frac{1}{x}+\sum_1^\infty(-)^{r-1}\,.\,\frac{2x}{r^2\pi^2-x^2}$, an absolutely convergent series;

$=\frac{1}{x}+\frac{1}{\pi-x}-\frac{1}{\pi+x}-\frac{1}{2\pi-x}+\frac{1}{2\pi+x}+\ldots$, a semi-convergent series; for all real values of x, except $x=\pm r\pi$.

II. When n is even,

$$\sec n\theta = \sum_{1}^{\frac{n}{2}} \frac{A_r}{\sin^2\frac{(2r-1)\pi}{2n} - \sin^2\theta},$$

where $A_r = \left[\dfrac{\sin^2\frac{(2r-1)\pi}{2n} - \sin^2\theta}{\cos n\theta}\right]_{\theta=\frac{(2r-1)\pi}{2n}}$

$$= \frac{2\sin\frac{(2r-1)\pi}{2n}\cos\frac{(2r-1)\pi}{2n}}{n\sin\frac{(2r-1)\pi}{2}} = \frac{(-)^{r-1}}{n}.\sin\frac{(2r-1)\pi}{n};$$

$$\therefore\ \sec n\theta = \frac{1}{n}\sum_{1}^{\frac{n}{2}}(-)^{r-1}.\frac{\sin\frac{(2r-1)\pi}{n}}{\sin^2\frac{(2r-1)\pi}{2n} - \sin^2\theta};$$

[and when n is odd,

$$\cos\theta.\sec n\theta = \frac{1}{n}\sum_{1}^{\frac{n-1}{2}}(-)^{r-1}.\frac{\sin\frac{(2r-1)\pi}{n}.\cos\frac{(2r-1)\pi}{2n}}{\sin^2\frac{(2r-1)\pi}{2n} - \sin^2\theta}].$$

Putting $\theta = \frac{x}{n}$, we get

$$\sec x = \frac{1}{n}\sum_{1}^{k}(-)^{r-1}.\frac{\sin\frac{(2r-1)\pi}{n}}{\sin^2\frac{(2r-1)\pi}{2n} - \sin^2\frac{x}{n}} + (-)^k.R$$

and, as before, if n is so great that $(2k-1)$ can be taken greater than $\frac{2x}{\pi}$

$$R = \frac{\epsilon}{n}.\frac{\sin\frac{(2k+1)\pi}{n}}{\sin^2\frac{(2k+1)\pi}{2n} - \sin^2\frac{x}{n}},\ \text{for}$$

$$\sin\frac{(2r-1)\pi}{n}\left\{\sin^2\frac{(2r+1)\pi}{2n} - \sin^2\frac{x}{n}\right\}$$

$$-\sin\frac{(2r+1)\pi}{n}\left\{\sin^2\frac{(2r-1)\pi}{2n} - \sin^2\frac{x}{n}\right\}$$

$$=2\sin\frac{\pi}{n}\left\{\sin\frac{(2r-1)\pi}{2n}\cdot\sin\frac{(2r+1)\pi}{2n}+\sin^2\frac{x}{n}\cdot\cos\frac{2r\pi}{n}\right\}$$

$$>2\sin\frac{\pi}{n}\left\{\sin^2\frac{(2r-1)\pi}{2n}-\sin^2\frac{x}{n}\right\},$$

and $\therefore$ the terms of R are in descending order of magnitude.

Hence, $\sec x$

$$=\frac{1}{n}\sum_1^k(-)^{r-1}\cdot\frac{\sin\frac{(2r-1)\pi}{n}}{\sin^2\frac{(2r-1)\pi}{2n}-\sin^2\frac{x}{n}}+(-)^k\cdot\frac{\epsilon}{n}\cdot\frac{\sin\frac{(2k+1)\pi}{n}}{\sin^2\frac{(2k+1)\pi}{2n}-\sin^2\frac{x}{n}}$$

and proceeding to the limit, we get, exactly as in I,

$$\sec x=4\pi\sum_1^\infty(-)^{r-1}\cdot\frac{2r-1}{(2r-1)^2\pi^2-4x^2},\text{ a semi-convergent series};$$

$$=2\left\{\frac{1}{\pi-2x}+\frac{1}{\pi+2x}-\frac{1}{3\pi-2x}-\frac{1}{3\pi+2x}+\dots\dots\right\};$$

for all real values of x except $x=\pm\frac{(2r-1)\pi}{2}$.

III. When n is even

$$\sec\theta\,.\cot n\theta=\frac{\prod_1^{\frac{n}{2}}\left\{\sin^2\frac{(2r-1)\pi}{2n}-\sin^2\theta\right\}}{2^{n-1}.\sin\theta(1-\sin^2\theta)\prod_1^{\frac{n}{2}-1}(\sin^2\frac{r\pi}{n}-\sin^2\theta)};$$

$$\therefore\quad\frac{\cos n\theta}{\sin\theta\cos\theta\sin n\theta}=\sum_0^{\frac{n}{2}}\frac{A_r}{\sin^2\frac{r\pi}{n}-\sin^2\theta},$$

$$\text{where }A_0=\left[-\frac{\sin\theta\cos n\theta}{\cos\theta\sin n\theta}\right]_{\theta=0}=-\frac{1}{n},$$

$$A_r=\left[\frac{\cos n\theta\left(\sin^2\frac{r\pi}{n}-\sin^2\theta\right)}{\sin\theta\cos\theta\,.\sin n\theta}\right]_{\theta=\frac{r\pi}{n}}=-\frac{2}{n},$$

if $r=1,\ 2,\ \dots\dots,\ \frac{n}{2}-1,$

$$\text{and } A_{\frac{n}{2}} = \left[\frac{\cos n\theta\cos\theta}{\sin\theta\sin n\theta}\right]_{\theta=\frac{\pi}{2}} = -\frac{1}{n};$$

$$\therefore\ \sec\theta\cot n\theta = \frac{1}{n\sin\theta} - \frac{2\sin\theta}{n}\sum_{1}^{\frac{n}{2}-1}\frac{1}{\sin^2\frac{r\pi}{n} - \sin^2\theta} - \frac{\sin\theta}{n\cos^2\theta}.$$

[When n is odd,

$$\sec\theta\cot n\theta = \frac{1}{n\sin\theta} - \frac{2\sin\theta}{n}\sum_{1}^{\frac{n-1}{2}}.\frac{1}{\sin^2\frac{r\pi}{n} - \sin^2\theta}].$$

Putting $\theta = \frac{x}{n}$, we get

$$\sec\frac{x}{n}\cot x$$

$$= \frac{1}{n\sin\frac{x}{n}} - \frac{\sin\frac{x}{n}}{n\cos^2\frac{x}{n}} - 2n\sin\frac{x}{n}\sum_{1}^{k}\frac{1}{n^2\left\{\sin^2\frac{r\pi}{n} - \sin^2\frac{x}{n}\right\}} - 2n\sin\frac{x}{n}.\,R,$$

$$\text{where } R = \sum_{k+1}^{\frac{n}{2}-1}\frac{1}{n^2\left\{\sin^2\frac{r\pi}{n} - \sin^2\frac{x}{n}\right\}}$$

$$< \sum_{k+1}^{\frac{n}{2}-1}\frac{1}{n^2\left\{\frac{4r^2}{n^2} - \frac{x^2}{n^2}\right\}},\ \text{since } \phi > \sin\phi > \frac{2\phi}{\pi} \text{ if } 0 < \phi < \frac{\pi}{2};$$

provided that n is so great that $2k$ can be taken greater than x.

$$\therefore\ R < \sum_{k+1}^{\frac{n}{2}-1}\frac{1}{4r^2 - x^2} < \sum_{k+1}^{\infty}\frac{1}{4r^2 - x^2},$$ the remainder after k terms of a convergent infinite series.

Proceeding to the limit

$$\cot x = \frac{1}{x} - \sum_{1}^{k}\frac{2x}{r^2\pi^2 - x^2} - 2x.\,R_1;$$

and R_1, the limiting value of R, can be made as small as we please by choosing k great enough.

$$\therefore \cot x = \frac{1}{x} - \sum_{1}^{\infty} \frac{2x}{r^2\pi^2 - x^2} \text{ an absolutely convergent series;}$$

$$= \frac{1}{x} - \frac{1}{\pi - x} + \frac{1}{\pi + x} - \frac{1}{2\pi - x} + \frac{1}{2\pi + x} - \ldots\ldots, \text{ a semi-conver-}$$

gent series; for all real values of x, except $x = \pm r\pi$.

IV. When n is even,

$$\sec\theta \operatorname{cosec}\theta \,.\, \tan n\theta = \sum_{1}^{\frac{n}{2}} \frac{A_r}{\sin^2\frac{(2r-1)\pi}{2n} - \sin^2\theta},$$

$$\text{where } A_r = \left[\frac{\sin n\theta\left\{\sin^2\frac{(2r-1)\pi}{2n} - \sin^2\theta\right\}}{\sin\theta\cos\theta\cos n\theta}\right]_{\theta=\frac{(2r-1)\pi}{2n}} = \frac{2}{n};$$

$$\therefore \tan n\theta = \frac{2\sin\theta\cos\theta}{n} \sum_{1}^{\frac{n}{2}} \frac{1}{\sin^2\frac{(2r-1)\pi}{2n} - \sin^2\theta}.$$

[When n is odd,

$$\tan n\theta = \frac{1}{n}\tan\theta + \frac{2\sin\theta\cos\theta}{n} \sum_{1}^{\frac{n-1}{2}} \frac{1}{\sin^2\frac{(2r-1)\pi}{2n} - \sin^2\theta}].$$

Hence, exactly as in III.,

$$\tan x = \sum_{1}^{\infty} \frac{8x}{(2r-1)^2\pi^2 - 4x^2}, \text{ an absolutely convergent series;}$$

$$= 2\left\{\frac{1}{\pi - 2x} - \frac{1}{\pi + 2x} + \frac{1}{3\pi - 2x} - \frac{1}{3\pi + 2x} \cdots\right\}, \text{ a semi-con-}$$

vergent series; for all real values of x, except $x = \pm\frac{(2r-1)\pi}{2}$.

The continuity of the algebraic expressions ensures that these results still hold good when x has complex values.

Note on the method of finding the particular integral of the differential Equation

$$f(D)y = \sum_1^n a_r x^r.$$

By D. K. Picken.

The use of inverse operators is only justifiable when it is obvious what direct operation of the calculus they symbolise.

The purpose of this note is to point out how the usual method of obtaining the integral of this differential equation can be shown as the result of direct operations.

We suppose that

$$f(z) = A_0 + A_1 z + \dots + A_m z^m, \text{ where } A_0,\ A_1 \dots A_m \text{ are constants,}$$

$$= \frac{1}{\phi(z)}.$$

Let $\psi(z) = \phi(0) + z\phi'(0) + \dots + \dfrac{z^n}{n!}\phi^{(n)}(0)$, a polynomial of degree n;

then it is easy to show by Leibnitz's Theorem, if it is not otherwise obvious, that

$$f(z).\psi(z) = 1 + z^{n+1}.R_{m-1}(z), \text{ where } R_{m-1}(z)$$

is a polynomial of degree $(m-1)$;

$$\therefore\ f(D)y = \sum_1^n a_r x^r$$

$$= [f(D).\psi(D)] \sum_1^n a_r x^r;$$

i.e. $y = \psi(D).\sum_1^n a_r x^r$, which is the usual rule.

Inequality Theorem regarding the lines joining corresponding vertices of two equilateral, or directly similar, triangles.

By Charles Tweedie, M.A., B.Sc.

§ 1. *Theorem.*—If ABC, A′B′C′ are any two equilateral triangles in a plane, their vertices being taken in the same sense of rotation, of the three lines AA′, BB′, CC′, the sum of any two is not less than the third.

Let BC and B′C′ meet in P, CA and C′A′ in Q, AB and A′B′ in R (Fig. 2). Let a be the angle between corresponding sides. Then, in virtue of the common angle a, the quadrilaterals

AA′QR, BB′PR, CC′PQ are cyclic.

Hence $AA'/QR = BB'/PR = CC'/PQ = \sin a/\sin 60°$.

i.e., AA′, BB′, CC′ are proportional to the sides of the triangle PQR.

∴ etc. Q.E.D.

Cor. 1. Since the triangles are equilateral we may make B′ or C′ correspond to A, and the theorem therefore applies to

AA′, BB′, CC′ ;

AB′, BC′, CA′ ;

AC′, BA′, CB′ .

Cor. 2. If the triangle A′B′C′ become infinitesimal the theorem still applies ; or, if P be any point in the plane of an equilateral triangle ABC, of the three lines PA, PB, PC, the sum of any two is not less than the third.

§ 2. If, instead of being equilateral, the triangles ABC and A′B′C′ are only directly similar, the same construction as before leads to the relations

$$AA'/QR = \sin a/\sin A,$$
$$BB'/PR = \sin a/\sin B,$$
$$CC'/PQ = \sin a/\sin C.$$

Hence $\quad AA'\sin A, \quad BB'\sin B, \quad CC'\sin C,$

and therefore aAA', bBB', cCC', are proportional to the sides of the triangle PQR.

In particular, if P is a point in the plane of triangle ABC, then aPA, bPB, cPC are proportional to the sides of a triangle.

§ 3. The case of equality of one of the quantities in question to the sum of the other two will in general only arise when P, Q, R are collinear points. Now, when two triangles are directly similar, a point O can be found in the plane such that, by rotation and similarity transformation, the one triangle can be transformed into the other.

When the triangles are also in perspective, the point O is common to the two circumcircles of ABC and A′B′C′.*

For, let the circles BRPB′, CQPC′ cut in O (Fig. 3).

$$\therefore \quad \angle BOP + \angle BRP = \pi,$$

$$\angle COP + \angle CQP = \pi.$$

Therefore $$\angle BOP - \angle COP = \angle CQP - \angle BRP,$$

$$\therefore \quad \angle BOC = \angle RAQ = \angle BAC,$$

i.e., O is on the circle round ABC, and ∴ also on the circle A′B′C′.

Let, as before, α be the angle between corresponding sides. If α is kept fixed, the angle AOA′ is constant. Also OA/OA′ is constant. Hence when △ABC is kept fixed, and △A′B′C′ moves so that the case of equality arises, the loci of A′, B′, C′ are equal circles through A, B, C respectively. These circles vary when α is changed. Also, when A′B′C′ reduces to a point P, there is equality of one of aPA, bPB, cPC to the sum of the other two if P lies on the circumcircle of ABC.

The case of two similar and similarly situated triangles may be considered as arising from $\alpha = 0$ or $\alpha = \pi$, and the case of inequality is therefore the general case when P, Q, R are at infinity. If this is not clear, a particular case of the next paragraph will render the conclusion sufficiently obvious.

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* See note by Mr P. Pinkerton upon this paper, which suggested this conclusion.

§ 4. The theorem just discussed leads to an interesting analytical conclusion.

If X, Y, Z are three positive quantities, then of the three quantities

$$X+Y-Z, \quad X-Y+Z, \quad -X+Y+Z,$$

two are certainly positive; and if X be the greatest, there is only uncertainty as to the sign of $Y+Z-X$. Hence the sign of $Y+Z-X$ is the same as that of $(X+Y+Z)\Pi(X+Y-Z)$, *i.e.*, of $2\Sigma Y^2Z^2-\Sigma X^4$, and if this expression is always positive, so is each of the quantities $X+Y-Z$, etc.

Consider the coordinates of the vertices of two directly similar triangles ABC and A'B'C' to be fixed as follows. Let (ξ, η) denote the vertex C, α the inclination of CB to the x-axis, θ the angle BCA, a and b the sides CB and CA. Let (x, y), β, θ, ρa and ρb, be the corresponding elements for A'B'C'.

If ABC be fixed, and ρ and β fixed, then the equation

$$a\text{AA}'+b\text{BB}'-c\text{CC}'=0$$

would in general lead to a locus for C' which is a bicircular quartic, which should divide the plane into regions for which

$$a\text{AA}'+b\text{BB}'>c\text{CC}', \text{ and } a\text{AA}'+b\text{BB}'<c\text{CC}'.$$

But we already know that the latter case can not arise. Hence the locus must, if real and finite both ways, be represented by the square of a quadratic function of x and y, and therefore of a circular function of x and y, or of a quadratic point function multiplied by the square of a linear function of x and y. So long as a, b, and θ are distinct from zero the latter case can not arise. We are therefore led to expect that the analytic expression of the fact that the sum of any two of $a\text{AA}'$, $b\text{BB}'$, $c\text{CC}'$ is not less than the third leads to the conclusion that the expression $2\Sigma b^2c^2\text{BB}'^2.\text{CC}'^2-\Sigma a^4\text{AA}'^4$ when represented in terms of x, y, etc., is the square of a circular quadratic function of x and y.

The coordinates of C, B, A are

$$(\xi, \eta); \quad (\xi+a\cos\alpha, \ \eta+a\sin\alpha); \quad (\xi+b\cos\overline{\theta+\alpha}, \ \eta+b\sin\overline{\theta+\alpha}).$$

The coordinates of C′, B′, A′ are

$$(x,\ y)\ ;\ \ (x+\rho a\cos\beta,\ y+\rho a\sin\beta)\ ;\ \ (x+\rho b\cos\overline{\theta+\beta},\ y+\rho b\sin\overline{\theta+\beta}).$$

$$\therefore\quad AA'^2=(x-\xi)^2+(y-\eta)^2,$$

$$BB'^2=(x-\xi+\rho a\cos\beta-a\cos a)^2+(y-\eta+\rho a\sin\beta-a\sin a)^2,$$

$$CC'^2=(x-\xi+\rho b\cos\overline{\theta+\beta}-b\cos\overline{\theta+a})^2+(y-\eta+\rho b\sin\overline{\theta+\beta}-b\sin\overline{\theta+a})^2.$$

The expression

$$2\Sigma b^2c^2BB'^2.\,CC'^2-\Sigma a^4AA'^4$$

may then be proved equal to

$$4a^2b^2\{(\overline{x-\xi}^2+\overline{y-\eta}^2)\sin\theta-(x-\xi)(a\sin\overline{\theta+a}-b\sin a-\rho a\sin\overline{\theta+\beta}+\rho b\sin\beta)$$
$$+(y-\eta)(a\cos\overline{\theta+a}-b\cos a-\rho a\cos\overline{\theta+\beta}+\rho b\cos\beta)\}^2.$$

This result naturally includes the whole of the preceding theory. The labour of verification may be simplified by writing X for $x-\xi$ and Y for $y-\eta$, and replacing c^2 wherever it occurs by its equivalent $a^2+b^2-2ab\cos\theta$.

By taking $a=0$, $\beta=0$ or π, we obtain the case of figures similar and similarly situated, and there is equality for

$$(\overline{x-\xi}^2+\overline{y-\eta}^2)\sin\theta-(x-\xi)a\sin\theta(1\pm\rho)+(y-\eta)(a\cos\theta-b)(1\pm\rho)=0.$$

Hence if the sides of two congruent triangles ($\rho=1$) are like directed there can only be equality when $(x-\xi)^2+(y-\eta)^2=0$, *i.e.*, when two corresponding vertices are coincident. In all other cases the inequality holds good.

§ 5. The inequality theorem admits of extension to two similar triangles in parallel planes, when there is no case of equality, so long as the planes are distinct.

The most interesting case, that of two equilateral triangles, is also the simplest to discuss.

Let the distance between the two planes be d and project A′B′C′ upon the plane of ABC into $A_1B_1C_1$.

Let α, β, γ denote AA′, BB′, CC′,
and α_1, β_1, γ_1 AA_1, BB_1, CC_1.

$\therefore\quad \alpha^2 = d^2 + \alpha_1^2;\quad \beta^2 = d^2 + \beta_1^2;\quad \gamma^2 = d^2 + \gamma_1^2.$

$$\therefore\quad 2\Sigma\beta^2\gamma^2 - \Sigma\alpha^4 = 2\Sigma(d^2+\beta_1^2)(d^2+\gamma_1^2) - \Sigma(d^2+\alpha_1^2)^2$$
$$= 3d^4 + 2d^2\Sigma\alpha_1^2 + (2\Sigma\beta_1^2\gamma_1^2 - \Sigma\alpha_1^4)$$
$$= \text{a sum of positive quantities}, \quad \therefore \text{ etc.}$$

In the case of directly similar triangles we must prove

$$2\Sigma b^2c^2(d^2+\beta_1^2)(d^2+\gamma_1^2) - \Sigma a^4(d^2+\alpha_1^2)^2 > 0$$

i.e., $d^4(2\Sigma b^2c^2 - \Sigma a^4) + 2d^2\{\Sigma b^2c^2(\beta_1^2+\gamma_1^2) - \Sigma a^4\alpha_1^2\}$
$+ (2\Sigma b^2c^2\beta_1^2\gamma_1^2 - \Sigma a^4\alpha_1^4) > 0.$

The first and third members of this inequality are already positive and the theorem is true provided

$$\Sigma b^2c^2(\beta_1^2+\gamma_1^2) - \Sigma a^4\alpha_1^2 > 0,$$

i.e., $\Sigma a^2\alpha_1^2(b^2+c^2-a^2) > 0$

or $\Sigma a^2\alpha_1^2 .\, 2bc\cos A > 0,$

i.e., $abc\,\Sigma a\cos A\alpha_1^2 > 0$

or $\Sigma \alpha_1^2 \sin 2A > 0.$

The last statement is obvious for an acute-angled triangle. For a general proof, determine O and O_1 the circumcentres of the two triangles. Join OO_1, and the centres to the corresponding vertices.

Let OA make an angle ψ with $\overrightarrow{OO_1}$, and OA_1 an angle ψ_1 with $\overrightarrow{OO_1}$. Then if r and r_1 be the radii and d the distance OO_1, we have

$$AA_1^2 = d^2 + r^2 + r_1^2 - 2rr_1\cos(\psi - \psi_1) - 2d(r\cos\psi - r_1\cos\psi_1).$$

To get BB_1^2 put $\psi + 2C$ for ψ and $\psi_1 + 2C$ for ψ_1;

$\therefore\ BB_1^2 = d^2 + r^2 + r_1^2 - 2rr_1\cos(\psi - \psi_1) - 2d(r\cos\overline{\psi + 2C} - r_1\cos\overline{\psi_1 + 2C}).$

Similarly CC_1^2 is found by adding $2C + 2A$ to ψ and ψ_1.

Hence $\Sigma\alpha_1^2\sin 2A = (d^2 + r^2 + r_1^2 - 2rr_1\cos\overline{\psi - \psi_1})\Sigma\sin 2A + 0,$

(for the cofficients of dr and dr' are zero)

$$= (d^2 + r^2 + r_1^2 - 2rr_1\cos\overline{\psi - \psi_1}) \times 4\sin A\sin B\sin C,$$

and $\therefore\ \Sigma\alpha_1^2\sin 2A$ is always positive. $\therefore$ etc. Q.E.D.

Note on Mr Tweedie's Theorem in Geometry.

By Peter Pinkerton, M.A.

Let ABC, A′B′C′ (Fig. 4) be two triangles equiangular in the same sense. Let BC, B′C′ meet in X. Describe circles round BXB′, CXC′ to meet again in O. Then it is easy to see that the triangles BOC, COA, AOB are equiangular in the same sense to the triangles B′OC′, C′OA′, A′OB′ respectively. Hence the triangles AOA′, BOB′, COC′ are similar;

$$\therefore \quad \frac{AA'}{AO} = \frac{BB'}{BO} = \frac{CC'}{CO};$$

$\therefore$ $a \,.\, AA'$, $b \,.\, BB'$, $c \,.\, CC'$ are proportional to $a \,.\, AO$, $b \,.\, BO$, $c \,.\, CO$, where a, b, c are the sides of the triangle ABC.

From O draw OP, OQ, OR perpendicular to BC, CA, AB respectively.

$$\text{Then} \quad QR = AO \sin A \propto a \,.\, AO,$$
$$RP = BO \sin B \propto b \,.\, BO,$$
$$PQ = CO \sin C \propto c \,.\, CO;$$

$\therefore$ $a \,.\, AA'$, $b \,.\, BB'$, $c \,.\, CC'$, being proportional to $a \,.\, AO$, $b \,.\, BO$, $c \,.\, CO$, are proportional to QR, RP, PQ.

But PQR is a triangle, unless O is on the circumcircle of ABC when PQR is the Simson line of O.

$\therefore$ $QR + RP > PQ$, with two similar inequalities, except that *one* of the inequalities becomes an equality if O is on the circumcircle of ABC.

$\therefore$ $a \,.\, AA' + b \,.\, BB' > c \,.\, CC'$, with two similar inequalities; one of the inequalities becoming an equality when O lies on the circumcircle of ABC.

Similarily in the case of an equality O lies also on the circumcircle of A′B′C′.

For the case of equilateral triangles $a = b = c$;

$\therefore$ $AA' + BB' > CC'$, with two similar inequalities; one of the three inequalities becoming an equality when O lies on the circumcircles of ABC and A′B′C′.

It is obvious that the theorem reduces to Ptolemy's Theorem or its converse.

Fourth Meeting, 12th February 1904.

Mr Charles Tweedie, President, in the Chair.

A Basic-sine and cosine with symbolical solutions of certain differential equations.

By F. H. Jackson, M.A.

The object of this paper is to introduce certain functions analogous to the circular functions. The functions will be denoted by

$$\sin_p(\lambda, x), \ \cos_p(\lambda, x).$$

Formulæ analogous to

$$\sin^2 a + \cos^2 a = 1,$$

$$\cos^2 a - \sin^2 a = \cos 2a,$$

$$\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b,$$

$$\cos(a \mp b) = \cos a \cos b \pm \sin a \sin b,$$

will be obtained, and the use of the functions in symbolical solutions of certain differential equations exemplified. The connection of the functions with generalised Bessel-functions of order half an odd integer will be shown.

1.

Consider the function

$$E_p(\lambda) = 1 + \frac{\lambda}{[1]!} + \frac{\lambda^2}{[2]!} + \dots + \frac{\lambda^r}{[r]!} + \dots, \qquad (1)$$

$$\lambda = 1 \text{ or } < 1,$$

in which $[r] = \dfrac{p^r - 1}{p - 1}$,

$$[r]! = [1][2][3]\dots[r].$$

If we invert the base p, we obtain

$$E_{\frac{1}{p}}(\lambda) = 1 + \frac{\lambda}{[1]!} + p\frac{\lambda^2}{[2]!} + \dots + p^{r.\,r-1/2}\frac{\lambda^r}{[r]!} + \dots. \qquad (2)$$

It is well known that

$$(1+\lambda)(1+p\lambda)(1+p^2\lambda)\dots(1+p^{m-1}\lambda)$$

$$=1+\sum_{r=1}^{r=m}\frac{(p^m-1)(p^{m-1}-1)(p^{m-2}-1)\dots(p^{m-r+1}-1)}{(p-1)(p^2-1)(p^3-1)\dots(p^r-1)}p^{r.r-1/2}\lambda^r, \quad (3)$$

$$(p<1).$$

When m is infinite this reduces to

$$(1+\lambda)(1+p\lambda)(1+p^2\lambda)\dots \textit{ ad inf.}$$

$$=1+\sum_{r=1}^{r=\infty}(-1)^r\frac{\lambda^r}{(p-1)(p^2-1)(p^3-1)\dots(p^r-1)}p^{r.r-1/2}. \quad (4)$$

If now for λ we substitute $\lambda(1-p)$,
we obtain

$$\{1+\lambda(1-p)\}\{1+p\lambda(1-p)\}\{1+p^2\lambda(1-p)\}\dots \textit{ ad inf.}$$

$$=1+\sum_{r=1}^{r=\infty}\frac{\lambda^r}{[r]!}p^{r.r-1/2} \quad - \quad - \quad - \quad - \quad (5)$$

$$=\mathrm{E}_{\frac{1}{p}}(\lambda).$$

Inverting the base p we obtain

$$\left\{1+\lambda\left(1-\frac{1}{p}\right)\right\}\left\{1+\frac{\lambda}{p}\left(1-\frac{1}{p}\right)\right\}\dots \textit{ ad inf.} \quad - \quad - \quad (6)$$

$$=1+\sum_{r=1}^{r=\infty}\frac{\lambda^r}{[r]!}$$

$$=\mathrm{E}_p(\lambda).$$

The infinite products are convergent only when $p<1$ and $p>1$ respectively, but the series have a much wider range of convergence, for, subject to limitations of the value of λ in $\mathrm{E}_{\frac{1}{p}}(\lambda)$, the series are convergent for all finite values of p.

2.

The expression (3) may be written

$$\frac{1}{[m]!}+\frac{a}{[m-1]![1]!}+p\frac{a^2}{[m-2]![2]!}+\dots\dots+p^{m.m-1/2}\frac{a^m}{[m]!}$$

$$\equiv\frac{(1+a)(1+pa)\dots\dots(1+p^{m-1}a)}{[m]!}. \quad - \quad - \quad - \quad (7)$$

Forming now the product

$$E_p(a) \,.\, E_{\frac{1}{p}}(b)$$

since the series are absolutely convergent we obtain the series

$$1+\left\{\frac{a}{[1]!}+\frac{b}{[1]!}\right\}+\left\{\frac{a^2}{[2]!}+\frac{ab}{[1]![1]!}+p\frac{b^2}{[2]!}\right\}+\ldots$$

$$+\left\{\frac{a^r}{[r]!}+\frac{a^{r-1}b}{[r-1]![1]!}+p\frac{a^{r-2}b^2}{[r-2]![2]!}+\ldots+p^{r\cdot r-1/2}\frac{b^r}{[r]!}\right\}+\ldots$$

$$=1+\frac{(a+b)}{[1]!}+\frac{(a+b)(a+pb)}{[2]!}+\frac{(a+b)(a+pb)(a+p^2b)}{[3]!}+\ldots \quad (8)$$

Putting $b=-a$ this gives us

$$E_p(a) \,.\, E_{\frac{1}{p}}(-a)=1, \quad - \quad - \quad - \quad - \quad (9)$$

analogous to

$$e^a \times e^{-a}=1.$$

Putting $b=a$ we obtain

$$E_p(a)E_{\frac{1}{p}}(a)=1+\frac{2a}{[1]!}+\frac{2(1+p)a^2}{[2]!}+\frac{2(1+p)(1+p^2)a^3}{[3]!}+\ldots \quad (10)$$

Considering numbers as formed from a sequence 1, p^0, p^1, p^2,...... we can show that the analogue of 2^2 is $(1+p^0)(1+p^1)$,

„ 2^3 „ $(1+p^0)(1+p^1)(1+p^2)$,

„ 3^2 „ $(1+p^0+p^1)(1+p^0+p^2)$,

....

We therefore write

$$E_p(a)E_{\frac{1}{p}}(a)=1+\frac{(2)_1a}{[1]!}+\frac{(2)_2a^2}{[2]!}+\ldots+\frac{(2)_ra^r}{[r]!}+\ldots \quad - \quad (11)$$

This idea of number can be extended to forms $\{x\}_n$ and $(x)_n$ analogous to x^n when n and x are not restricted to integral values.

3.

The functions $\sin_p(a)$, $\cos_p(a)$.

We define these as follows :—

$$\cos_p(a) = \frac{E_p(ia) + E_p(-ia)}{2}$$

$$= 1 - \frac{a^2}{[2]!} + \frac{a^4}{[4]!} - \ldots\ldots \qquad (12)$$

and

$$\sin_p(a) = \frac{E_p(ia) - E_p(-ia)}{2i}$$

$$= \frac{a}{[1]!} - \frac{a^3}{[3]!} + \frac{a^5}{[5]!} - \ldots\ldots \qquad (13)$$

From these forms we obtain directly

$$\sin_p(a)\sin_{\frac{1}{p}}(a) + \cos_p(a)\cos_{\frac{1}{p}}(a) = 1 \qquad (14)$$

and

$$\cos_p(a)\cos_{\frac{1}{p}}(a) - \sin_p(a)\sin_{\frac{1}{p}}(a) \qquad (15)$$

$$= 1 - \frac{(2)_2 a^2}{[2]!} + \frac{(2)_4 a^4}{[4]!} - \ldots\ldots,$$

where $\qquad (2)_r = (1+p^0)(1+p^1)(1+p^2)\ldots(1+p^{r-1}).$

This series is the analogue of the series for $\cos 2a$.

Now

$$\sin_p(a)\cos_{\frac{1}{p}}(b) = \frac{a}{[1]!} - \left\{\frac{a^3}{[3]!} + p\frac{ab^2}{[1]![2]!}\right\} + \ldots,$$

$$\cos_p(a)\sin_{\frac{1}{p}}(b) = \frac{b}{[1]!} - \left\{\frac{a^2 b}{[2]![1]!} + p^3\frac{b^3}{[3]!}\right\} + \ldots.$$

Therefore

$$\sin_p(a)\cos_{\frac{1}{p}}(b) + \cos_p(a)\sin_{\frac{1}{p}}(b)$$

$$= \frac{(a+b)}{[1]!} - \frac{(a+b)(a+pb)(a+p^2b)}{[3]!} + \ldots$$

$$= \mathfrak{S}_p(a, b). \qquad (16)$$

If we denote

$$1 - \frac{(a+b)(a+pb)}{[2]!} + \frac{(a+b)(a+pb)(a+p^2b)(a+p^3b)}{[4]!} - \ldots$$

by $\mathfrak{C}(a, b)$,

the formulae may be written

$$\sin_p(a)\cos_{\frac{1}{p}}(b) \pm \cos_p(a)\sin_{\frac{1}{p}}(b) = \mathfrak{S}(a, \pm b), \quad - \quad - \quad (17)$$

$$\cos_p(a)\cos_{\frac{1}{p}}(b) \pm \sin_p(a)\sin_{\frac{1}{p}}(b) = \mathfrak{C}(a, \mp b), \quad - \quad - \quad (18)$$

$$\{\cos_p(a) + i\sin_p(a)\}\{\cos_{\frac{1}{p}}(b) + i\sin_{\frac{1}{p}}(b)\} = \mathfrak{C}(a, b) + i\mathfrak{S}(a, b), \quad (19)$$

$$\frac{E_p(a) + E_p(-a)}{E_p(a) - E_p(-a)} = \frac{E_p(a)E_{\frac{1}{p}}(a) + 1}{E_p(a)E_{\frac{1}{p}}(a) - 1} \quad - \quad - \quad - \quad (20)$$

$$= \frac{\mathfrak{E}(2, a) + 1}{\mathfrak{E}(2, a) - 1},$$

$\mathfrak{E}(2, a)$ denoting $1 + \frac{(2)_1 a}{[1]!} + \frac{(2)_2 a^2}{[2]!} + \ldots.$

Example:—

$$\frac{1}{[x]} + \frac{1}{[x][x+1]} + \frac{1}{[x][x+1][x+2]} + \ldots$$

$$= E_p(1)\left\{\frac{1}{[x]} - \frac{p}{[1]![x+1]} + \frac{p^3}{[2]![x+2]} - \ldots\right\} \quad (21)$$

The series

$$\frac{1}{[x]} + \frac{1}{[x][x+1]} + \ldots$$

can be expressed as the sum of a number of partial fractions

$$\frac{a_0}{[x]} + \frac{a_1}{[x+1]} + \frac{a_2}{[x+2]} + \ldots\ldots + \frac{a_n}{[x+n]} + \ldots$$

To find the coefficients a, multiply by $[x+n]$

and put $x = -n$; we thus obtain

$$a_n = \frac{1}{[-n][-n+1]\ldots[-3][-2][-1]}\left\{1 + \frac{1}{[1]!} + \frac{1}{[2]!} + \ldots\right\}$$

$$= (-1)^n \frac{p^{n.\,n+1/2}}{[n]!} E_p(1).$$

The required result is established.

In a similar way we may establish

$$\frac{p^x}{[x]}+\frac{p^{x+1}}{[x][x+1]}+\ldots\ldots+\frac{p^{rx+(r\,.\,r-1)/2}}{[x][x+1]\ldots[x+r-1]}+\ldots$$

$$=\mathrm{E}_{\frac{1}{p}}(p)\left\{\frac{1}{[x]}-\frac{1}{[1]\,!\,[x+1]}+\frac{1}{[2]\,!\,[x+2]}-\ldots\right\}. \quad (22)$$

In terms of the generalised Gamma-function * Γ_p, we may write these

$$\frac{1}{\Gamma_p([x+1])}+\frac{1}{\Gamma_p([x+2])}+\ldots=\frac{\mathrm{E}_p(1)}{\Gamma_p([x])}\left\{\frac{1}{[x]}-\frac{p}{[1]\,!\,[x+1]}+\ldots\right\}, \quad (23)$$

$$\frac{p^x}{\Gamma_p([x+1])}+\frac{p^{x+1}}{\Gamma_p([x+2])}+\ldots=\frac{\mathrm{E}_{1/p}(p)}{\Gamma_p([x])}\left\{\frac{1}{[x]}-\frac{1}{[1]\,!\,[x+1]}+\ldots\right\}, \quad (24)$$

both analogous to a well-known result

$$\frac{1}{\Gamma(x+1)}+\frac{1}{\Gamma(x+2)}+\ldots=\frac{e}{\Gamma(x)}\left\{\frac{1}{x}-\frac{1}{1\,!\,x+1}+\ldots\right\}.$$

4.

If we denote the convergent series

$$1+\frac{\lambda x^{[1]}}{[1]\,!}+\frac{\lambda^2 x^{[2]}}{[2]\,!}+\ldots \quad - \quad (25)$$

by $\mathrm{E}_p(\lambda,\ x)$,

then $$\frac{d}{dx}\,.\,\mathrm{E}_p(\lambda,\ x)=\lambda\mathrm{E}_p(\lambda,\ x^p),$$

$$\frac{d^{[n]}}{dx^{[n]}}\mathrm{E}_p(\lambda,\ x)=\lambda^n\mathrm{E}_p(\lambda,\ x^{p^n}),$$

$\mathrm{D}^{[n]}$ denoting $\frac{d}{d(x^{p^{n-1}})}\left\{\frac{d}{d(x^{p^{n-2}})}\left\{\ldots\left\{\frac{d}{d(a^{p^2})}\left\{\frac{d}{d(x^p)}\left\{\frac{d}{dx}\right\}\right\}\right\}\ldots\right\}\right.$.

If $$\mathrm{P}_p(x)=\int_0^1 \mathrm{E}_p(1,\ z^{p-x})\,.\,z^{p[x-1]}dz \quad - \quad (26)$$

then $$\mathrm{P}_p(x+1)=\frac{1}{p^x}[x]\,\mathrm{P}_p(x)-\frac{1}{\mathrm{E}_{\frac{1}{p}}(p)} \quad - \quad (27)$$

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* *Transactions, Royal Society, Edin.*, Vol. XLI., Art. 1.

which may also be written

$$P_p(x+1)+[-x]P_p(x)+E_p(-p)=0. \qquad (28)$$

Taking

$$\cos_p(\lambda, x)=\frac{E_p(i\lambda, x)+E_p(-i\lambda, x)}{2}, \qquad (29)$$

$$\sin_p(\lambda, x)=\frac{E_p(i\lambda, x)-E_p(-i\lambda, x)}{2i}, \qquad (30)$$

$$\frac{d^{[n]}}{dx^{[n]}}\left\{\sin_p(\lambda, x)\right\}=(-1)^{\frac{n-1}{2}}\lambda^n\cos_p(\lambda, x^{p^n}), \quad (n \text{ odd}),$$

$$=(-1)^{\frac{n}{2}}\ \lambda^n\sin_p(\lambda, x^{p^n}), \quad (n \text{ even}).$$

There must be a kind of periodicity for these functions analogous to that of the circular functions.

5.

Symbolic Solutions.

In the following analysis the functions $\sin_p$ and $\cos_p$ will be used to form symbolical solutions of the differential equation

$$p^{2n+2}\frac{d^{[2]}F}{dx^{[2]}}+\frac{[2n+2]}{x^p}\frac{dF}{dx}+\lambda^2 F(x^{p^2})=0. \qquad (31)$$

When the base $p=1$, this differential equation reduces to

$$\frac{d^2F}{dx^2}+\frac{2(n+1)}{x}\frac{dF}{dx}+\lambda^2F=0,$$

an equation of great interest in physical investigations (Lamb's *Hydrodynamics*, Arts. 267–309). Various solutions are

$$\psi_{[n]}(x)=(-1)^n\left(\frac{d}{xdx}\right)^n\frac{\sin\lambda x}{\lambda x},$$

$$\Psi_n(x)=(-1)^n\left(\frac{d}{xdx}\right)^n\frac{\cos x\lambda}{x\lambda},$$

$$f_n(x)=(-1)^n\left(\frac{d}{xdx}\right)^n\frac{e^{-ix\lambda}}{x\lambda}.$$

If we integrate the equation

$$p^{2n+2}\frac{d^{[2]}\mathrm{F}}{dx^{[2]}}+\frac{[2n+2]}{x^{p}}\frac{d\mathrm{F}}{dx}+\lambda^{2}\mathrm{F}(x^{p^{2}})=0$$

in series (*Proc. Edin. Math. Soc.*, Vol. XXI., pp. 65 et seq.), we obtain two series which we may denote

$$\psi_{[n]}(\lambda,\ x)=\frac{1}{[1][3][5]..[2n+1]}\left\{1-\frac{\lambda^{2}x^{[2]}}{[2][2n+3]}+\frac{\lambda^{4}x^{[4]}}{[2][4][2n+3][2n+5]}-..\right\} \quad - \quad (32)$$

$$\Psi_{[n]}(\lambda,\ x)=\frac{[1][3][5]..[2n-1]}{x_{1}^{[2n+1]}}\cdot\frac{1}{pn^{2}}\left\{1-\frac{\lambda^{2}x_{1}^{[2]}}{[2][1-2n]}+\frac{\lambda^{4}x_{1}^{[4]}}{[2][4][1-2n][3-2n]}-..\right\} \quad (33)$$

Operating with $\{\mathrm{D}^{[n]}\}$ on the series

$$\Sigma(-1)^{r}\frac{\lambda^{2r}x^{[2r]}}{[2r+1]}=\frac{\sin_{p}(\lambda,\ x^{\frac{1}{p}})}{\lambda x^{\frac{1}{p}}},$$

if $\left\{\mathrm{D}^{[n]}\right\}=\left\{\frac{1}{x^{p^{2n-1}}}\frac{d.}{d(x^{p^{2n-2}})}\left\{\cdots\cdots\left\{\frac{1}{x^{p^{3}}}\frac{d}{d(x^{p^{2}})}\left\{\frac{1}{x^{p}}\frac{d}{dx}\right\}\right\}\cdots\right\}\right\}$,

then all terms before

$$(-1)^{n}\frac{\lambda^{2n}x^{[2n]}}{[2n+1]!}$$

are destroyed, while the operations performed on the remaining terms of the series give us

$$(-1)^{n}\frac{\lambda^{2n}}{[1][3][5]..[2n+1]}\left\{1-\frac{\lambda^{2}x^{p^{2n}[2]}}{[2][2n+3]}+\frac{\lambda^{4}x^{p^{2n}[4]}}{[2][4][2n+3][2n+5]}-\cdots\right\}.$$

We see that

$$\psi_{[n]}(x^{p^{2n}},\ \lambda)=(-1)^{n}\lambda^{-2n}\{\mathrm{D}^{[n]}\}\cdot\frac{\sin_{p}(\lambda,\ x^{\frac{1}{p}})}{\lambda x^{\frac{1}{p}}}, \quad - \quad - \quad (34)$$

and by a change of the variable we may write this

$$\psi_{[n]}(\lambda,\ x)=(-1)^{n}\lambda^{-2n}\{\Delta^{[n]}\}\frac{\sin_{p}(\lambda,\ x^{p^{-1-2n}})}{\lambda x^{p^{-2-2n}}}, \quad - \quad - \quad (35)$$

where $\{\Delta^{[n]}\}$ is the operator $\left\{\frac{1}{x^{p^{-1}}}\frac{d}{d(x^{p^{-2}})}\left\{\cdots\left\{\frac{1}{x^{p^{1-2n}}}\frac{d}{dx^{p^{-2n}}}\right\}\right\}\right\}$.

In the same way

$$\Psi_{[n]}(\lambda, x) = (-1)^n \lambda^{-2n} \{\Delta^{[n]}\} \frac{\cos_p(\lambda, x^{p-1-2n})}{\lambda x^{p-1-2n}}, \quad \text{- - (36)}$$

$$\begin{aligned} f_{[n]}(\lambda, x) &= \lambda^{2n} \{\Psi_{[n]}(\lambda, x) - \psi_{[n]}(\lambda, x)\} \\ &= (-1)^n \{\Delta^{[n]}\} \frac{E_p(-i\lambda, x^{p-1-2n})}{\lambda x^{p-1-2n}}, \quad \text{- - (37)} \end{aligned}$$

may be established.

Recurrence-formula.

The recurrence-formula for the function $\psi_{[n]}$ may easily be established as

$$x\psi'_{[n]}\left(\frac{\lambda}{p}, x\right) + [2n+1]\psi_{[n]}(\lambda, x) = \psi_{[n-1]}(\lambda, x) \quad \text{- - (38)}$$

which may also be written in the form

$$-\frac{\lambda^2}{p^2} x^{[2]} \psi_{[n+1]}\left(x^{p^2}, \frac{\lambda}{p}\right) + [2n+1]\psi_{[n]}(\lambda, x) = \psi_{[n-1]}(\lambda, x). \quad (39)$$

6.

Connection with generalised Bessel functions of order half an odd integer.

We define $J_{[n]}(\lambda, x)$

as

$$\sum_{r=0}^{r=\infty} (-1)^r \frac{\lambda^{n+2r} x^{[n+2r]}}{[r]!\,[n+r]!\,(2)_r (2)_{n+r}}. \quad \text{- - - (40)}$$

The differential equation satisfied by this function may be obtained from (E) page 70, Vol. XXI., *Proc. Edin. Math. Soc.*, by the introduction of a parameter λ (*Trans. R. S. Edin.*, Vol. XLI.). In the following analysis theorems analogous to

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sin x,$$

$$J_{\frac{3}{2}}(x) = -\left(\frac{2x}{\pi}\right)^{\frac{1}{2}} \frac{d}{dx}\left(\frac{\sin x}{x}\right),$$

will be obtained :

$$J_{[n]}(\lambda, x) = \sum_{r=0}^{r=\infty} (-1)^r \frac{\lambda^{n+2r} x^{[n+2r]}}{[r]!\,[n+r]!\,(2)_r (2)_{n+r}},$$

in which $[n+r]! = \Gamma_p([n+r+1])$,

$$\Gamma_p([z+1]) = \underset{\kappa=\infty}{\mathrm{L}} \frac{[1][2][3][4]\ldots\ldots\ldots\ldots [\kappa]}{[z+1][z+2][z+3]\ldots\ldots[z+\kappa]}[\kappa]^z p^{\frac{z \cdot z+1}{2}},$$

$$(m)_z = (m)^z \frac{\Gamma_{p^z}([m+1])}{\Gamma_p([m+1])}.$$

These functions are considered in a paper shortly to be printed (*Proc. R. S. Lond.*). Here we only require the difference equations

$$\frac{1}{[x]} \times \Gamma_p([x+1]) = \Gamma_p([x]),$$

$$(2)_x = (p^x+1) \times (2)_{x-1}.$$

Consider

$$J_{[\frac{1}{2}]}(\lambda, x) = \frac{\lambda^{\frac{1}{2}} x^{[\frac{1}{2}]}}{[\frac{1}{2}]!(2)_{\frac{1}{2}}}$$

$$\left\{1 - \frac{\lambda^2 x^{p^{\frac{1}{2}}[2]}}{[1]!(2)_1[\frac{3}{2}](p^{\frac{3}{2}}+1)} + \frac{\lambda^4 x^{p^{\frac{1}{2}}[4]}}{[2]!(2)_2[\frac{3}{2}][\frac{5}{2}](p^{\frac{3}{2}}+1)(p^{\frac{5}{2}}+1)} - \ldots\right\}; \quad (41)$$

we see that the series on the right side of the above reduces by means of the difference equations (38) to the standard form (40).

Since $[1]!(2)_1 = [2]$,

$[2]!(2)_2 = [4][2]$,

$\ldots \quad \ldots$

$\ldots \quad \ldots$

and $[\frac{3}{2}] \times (p^{\frac{3}{2}}+1) = [3]$,

$\ldots \quad \ldots$

$\ldots \quad \ldots$

$$J_{[\frac{1}{2}]}(\lambda, x) = \frac{\lambda^{\frac{1}{2}} x^{[\frac{1}{2}]}}{[\frac{1}{2}]!(2)_{\frac{1}{2}}}\left\{1 - \frac{\lambda^2 x^{p^{\frac{1}{2}}[2]}}{[1][2][3]} + \frac{\lambda^4 x^{p^{\frac{1}{2}}[4]}}{[1][2][3][4][5]} - \ldots\right\} \quad (42)$$

$$= \frac{\lambda^{\frac{1}{2}} x^{[\frac{1}{2}]}}{[\frac{1}{2}]!(2)_{\frac{1}{2}}} \cdot \frac{\sin_p(\lambda, x^{p^{-\frac{1}{2}}})}{\lambda x^{p^{-\frac{1}{2}}}}$$

$$= \frac{\lambda^{-\frac{1}{2}} x^{[-\frac{1}{2}]}}{[\frac{1}{2}]^{\frac{1}{2}} \Gamma_{p^2}[(\frac{3}{2})]} \cdot \sin_p(\lambda, x^{p^{-\frac{1}{2}}}). \quad - \quad - \quad - \quad (43)$$

When $p=1$ the function Γ_{p^2} reduces to Euler's Gamma-function

$$\Gamma(\tfrac{3}{2}) = \tfrac{1}{2}\sqrt{\pi}.$$

There is no difficulty in extending

$$\mathrm{J}_{\frac{3}{2}}(x) = -\left(\frac{2x}{\pi}\right)^{\frac{1}{2}} \frac{d}{dx}\left(\frac{\sin x}{x}\right)$$

in the form

$$\mathrm{J}_{[\frac{3}{2}]}(\lambda, x) = -\frac{\lambda^{\frac{1}{2}} x^{[\frac{1}{2}]}}{[\frac{1}{2}]^{\frac{1}{2}}\Gamma_{p^2}([\frac{3}{2}])} \frac{d}{d(x^{p^{-\frac{1}{2}}})}\left\{\frac{\sin_p(\lambda, x^{p^{-\frac{3}{2}}})}{\lambda x^{p^{-\frac{3}{2}}}}\right\}; \quad - \quad (44)$$

and generally the formula

$$\left(\frac{\pi}{2x}\right)^{\frac{1}{2}} i^n \mathrm{J}_{n+\frac{1}{2}}(x) = \mathrm{P}_n\left(\frac{d}{idx}\right)\left(\frac{\sin x}{x}\right)$$

which is due to Lord Rayleigh (*Theory of Sound*, Vol. II., p. 263) may be extended to the functions

$$\mathrm{J}_{[n+\frac{1}{2}]}, \quad \mathrm{P}_{[n]}, \quad \sin_p, \qquad - \quad - \quad - \quad (45)$$

by means of the following identity

$$1 - \frac{[n][n-1}{[2][2n-1]} \cdot \frac{[2n+2m+1]}{[2n+2m-1]} + p^2 \frac{[n][n-1][n-2][n-3]}{[2][4][2n-1][2n-3]} \cdot \frac{[2n+2m+1]}{[2n+2m-3]} - \ldots$$

$$= p^{\frac{n.\,n-1}{2}} \frac{(2)_{n+m}(2)_n}{(2)_m} \frac{[n+m]!\,[n+2m]!\,[n]!\,[n]!}{[2n+2m]!\,[m]!\,[2n]!}. \quad (46)$$

The general term of the series is

$$(-1)^r p^{r.\,r-1} \frac{[n][n-1][n-2]\ldots\ldots\ldots\ldots[n-2r+1]}{[2][4]\ldots[2r].[2n-1]\ldots[2n-2r+1]} \cdot \frac{[2n+2m+1]}{[2n+2m-2r+1]}.$$

This identity is a particular case of summation of

$$\mathrm{F}_p([\alpha][\beta][\gamma][\delta][\epsilon])$$

and is a product of the two following series :— *

$$1 - p^2 \frac{[n][n-1]}{[2][2n-1]} + \ldots + (-1)^r p^{r.\,r+1} \frac{[n][n-1][n-2]\ldots[n-2r+1]}{[2][4]\ldots[2r].[2n-1]\ldots[2n-2r+1]} + \ldots$$

$$= \frac{[n]!\,[n]!\,(2)_n}{[2n]!}, \quad - \quad - \quad (47)$$

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* *Transactions R.S.E.*, Vol. XLI. "Generalised Functions of Legendre and Bessel," Part I. (57), Part II. (8).

$$1 - p^2\frac{[n][n-1]}{[2][2n+2m-1]}p^{2m} + p^4\frac{[n][n-1][n-2][n-3]}{[2][4][2n+2m-1][2n+2m-3]}p^{4m} - \dots$$

$$= p^{\frac{n.n-1}{2}}\frac{[n+m]!\,[n+2m]!\,(2)_{n+m}}{[2n+2m]!\,[m]!\,(2)_m}. \qquad (48)$$

Putting $m = 0$, $p = 1$,

we obtain the following, which I suppose is a known result:

$$\left\{1 - \frac{n.n-1}{2.2n-1} + \frac{n.n-1.n-2.n-3}{2.4.2n-1\;2n-3} - \dots\right\}^2 = 1 - \frac{n.n-1}{2.2n-1}\cdot\frac{2n+1}{2n-1}$$

$$+ \frac{n.n-1.n-2.n-3}{2.4.2n-1.2n-3}\cdot\frac{2n+1}{2n-3} - \dots; \qquad (49)$$

if c_1, c_2 ... be the coefficients in Legendre's series P_n

$$\{1 + c_1 + c_2 + \dots\}^2 = 1 + c_1\frac{2n+1}{2n-1} + c_2\frac{2n+1}{2n-3} + \dots. \qquad (50)$$

This, however, is outside the range of this paper, and must be left to a paper on $F([\alpha][\beta][\gamma][\delta][\epsilon])$.

Fifth Meeting, 11th March 1904.

Mr Charles Tweedie, President, in the Chair.

Some Points in Diophantine Analysis.

By Alexander Holm, M.A.

What I intend to put before you first is a graphical view of the solution of indeterminate equations of the third degree.

1. Suppose that a rational solution of an equation of the third degree in x and y is known, and that it is required to find other rational solutions.

For example, given $x=1$, $y=2$ a rational solution of

$$2x^3 - x^2y - 2xy^2 + y^3 + 7x^2 - xy - 2y^2 + 4x - 2y + 3 = 0,$$

to find other rational solutions.

Let the equation be represented by a graph (Fig. 5).

A straight line can be drawn through the given point (1, 2) to cut the graph in two more points; for there will arise a cubic equation to find the abscissae of the points of intersection.

Suppose now that the straight line is so drawn that one of these two points coincides with the given point, then the straight line becomes the tangent at the given point; and, the coordinates of the two coincident points being given rational, it follows that those of the other point of intersection are also rational.

Thus the tangent at the given point will cut the graph at a point whose coordinates are rational.

No new point will be obtained when the given point on the graph happens to be a point of inflexion.

Example: the above equation can be put into the form

$$(x - y + 2)(x + y + 1)(2x - y + 1) - 3x - y + 1 = 0.$$

Transfer the origin to the given point (1, 2) on the graph by putting

$$x = X + 1 \text{ and } y = Y + 2;$$

$$\therefore \quad (X - Y + 1)(X + Y + 4)(2X - Y + 1) - 3X - Y - 4 = 0.$$

The constant term is now zero, and the tangent at the new origin is

$$4(X-Y)+(X+Y)+4(2X-Y)-3X-Y=0,$$

that is, $Y=\frac{5}{4}X$, and it cuts the graph where

$$\left(-\frac{X}{4}+1\right)\left(\frac{9X}{4}+4\right)\left(\frac{3X}{4}+1\right)-3X-\frac{5X}{4}-4=0,$$

$$i.e.,\quad -\frac{27}{64}X^3+\frac{3}{8}X^2=0,$$

$$\therefore\quad X=0 \text{ or } 0 \text{ or } \frac{8}{9}.$$

Taking $X=\frac{8}{9},\ Y=\frac{10}{9},$

$$\therefore\quad x=\frac{17}{9},\ y=\frac{28}{9}.$$

The coordinates of the point where the tangent at $\left(\frac{17}{9},\frac{28}{9}\right)$ cuts the graph could now be found, and so on. In this way an infinite number of rational values of x and y could be found to satisfy the given equation.

Diophantine problems of the third degree generally present themselves under one of the following two more limited forms.

To find a value of x which will make

(1) $ax^3+bx^2+cx+d=$ a square $=y^2$,

or (2) $ax^3+bx^2+cx+d=$ a cube $=y^3$,

having given one value of x which does so.
The usual Diophantine method of solution is equivalent to the above tangent-method.

2. *When the graph happens to have one or three rational asymptotes.*

If a straight line drawn through a given point on the graph rotates until it becomes parallel to an asymptote, one point of intersection with the graph recedes to infinity. Hence the cubic equation, which determines the abscissae of the points of intersection, becomes a quadratic equation ; and, the abscissa of the given point being rational, it follows that the abscissa of the other point of intersection is also rational.

Example: an asymptote to the above graph is $x+y+1=0$;
a straight line through the point (1, 2) parallel to it is $y=-x+3$;
this line cuts the graph where

$$(2x-1)4(3x-2)-3x+x-3+1=0,$$
$$i.e., \quad 4x^2-5x+1=0,$$
$$\therefore \quad \left.\begin{array}{l} x=1 \\ y=2 \end{array}\right\} \text{ or } \left.\begin{array}{l} x=\frac{1}{4} \\ y=\frac{11}{4} \end{array}\right\};$$

$\therefore \quad x=\frac{1}{4}, \ y=\frac{11}{4}$ is a new solution.

Similarly a straight line through the point (1, 2) parallel to the second asymptote $x-y+2=0$ cuts the graph at

$$x=0, \quad y=1.$$

A straight line through the point (1, 2) parallel to the third asymptote $2x-y+1=0$ cuts the graph at

$$x=-1, \quad y=-2.$$

3. If a straight line moves parallel to an asymptote until it coincides with the asymptote, a second point of intersection with the graph recedes to infinity. Hence the cubic equation, which determines the abscissae of the points of intersection, becomes a simple equation giving a rational value for the abscissa of the point where the asymptote cuts the graph.
e.g., the asymptote $x+y+1=0$ cuts the graph at $x=1$, $y=-2$.

4. *When two rational solutions are known.*

A straight line joining two points with rational coordinates will cut the graph again in a third point, whose coordinates are rational. This third point will be different from either of the two given points, except when the straight line joining these two points happens to be the tangent at one of them. No third point of intersection will be obtained when the straight line joining the two given points happens to be parallel to an asymptote.
Thus (0, 1) and (-1, -2) are points on the graph.
The straight line joining them is $y=3x+1$, and it cuts the graph where

$$(-2x+1)(4x+2)(-x)-6x=0,$$
$$i.e., \quad 8x^3-8x=0;$$
$$\therefore \quad \left.\begin{array}{l} x=0 \\ y=1 \end{array}\right\} \text{ or } \left.\begin{array}{l} x=-1 \\ y=-2 \end{array}\right\} \text{ or } \left.\begin{array}{l} x=1 \\ y=4 \end{array}\right\};$$

$\therefore \quad x=1, \ y=4$ is a new solution.

I have not noticed any Diophantine methods corresponding to §§ 2, 3, 4.

5. *An example of the tangent-method.*

To find two cubes whose difference is equal to the sum of two given cubes.

Let a^3 and b^3 be the given cubes, x^3 and y^3 the required cubes,

then $$x^3 - y^3 = a^3 + b^3\,;$$

$$x = a, \qquad y = -b \text{ is a particular solution.}$$

Let $$x = \mathrm{X} + a, \; y = \mathrm{Y} - b\,;$$

$$\therefore \quad \mathrm{X}^3 - \mathrm{Y}^3 + 3a\mathrm{X}^2 + 3b\mathrm{Y}^2 + 3a^2\mathrm{X} - 3b^2\mathrm{Y} = 0.$$

Choose $$3a^2\mathrm{X} - 3b^2\mathrm{Y} = 0 \quad \text{or} \quad \mathrm{Y} = \frac{a^2}{b^2}\mathrm{X}.$$

$$\therefore \quad \left(1 - \frac{a^6}{b^6}\right)\mathrm{X}^3 + 3a\left(1 + \frac{a^3}{b^3}\right)\mathrm{X}^2 = 0.$$

$$\therefore \quad \mathrm{X} = 0 \text{ or } 0 \text{ or } \frac{3ab^3}{a^3 - b^3}.$$

Taking $$\mathrm{X} = \frac{3ab^3}{a^3 - b^3}, \qquad \mathrm{Y} = \frac{3a^3b}{a^3 - b^3},$$

$$\therefore \quad x = \frac{a^3 + 2b^3}{a^3 - b^3} \cdot a, \quad y = \frac{2a^3 + b^3}{a^3 - b^3} \cdot b\,.$$

Thus $$a^3 + b^3 = \left(\frac{a^3 + 2b^3}{a^3 - b^3} \cdot a\right)^3 - \left(\frac{2a^3 + b^3}{a^3 - b^3} \cdot b\right)^3;$$

e.g., if $a = 2$, $b = 1$ then $2^3 + 1^3 = \left(\frac{20}{7}\right)^3 - \left(\frac{17}{7}\right)^3$.

The problem of this article is the converse of Diophantus' third porism, which, along with the analogous problems added by Vieta and Bachet, can be resolved in a similar manner.*

* Cf. Heath's *Diophantus*, Camb. 1885, pp. 123–124.
Bachet's *Diophantus*, Paris 1621, pp. 178–182.
Oeuvres de Fermat, Paris 1891, t. I., pp. 297–299.

6. The sides of a rational right-angled triangle can be represented in the most general way possible by

$$(m^2+1)n, \ (m^2-1)n \text{ and } 2mn,$$

where m and n are positive rational numbers, m being >1.

If the sides are x, y, z, then $x^2=y^2+z^2$. (*Euc.* I., 47.)

$$\therefore \quad x^2-y^2=z^2;$$

$$\therefore \quad (x+y)(x-y)=z^2;$$

$$\therefore \quad \frac{x+y}{z}=\frac{z}{x-y}=\text{a rational number}=m;$$

$$\therefore \quad x+y-mz=0$$

$$\text{and} \quad -mx+my+z=0;$$

$$\therefore \quad \frac{x}{m^2+1}=\frac{y}{m^2-1}=\frac{z}{2m}=n;$$

$$\therefore \quad x=(m^2+1)n, \ y=(m^2-1)n, \ z=2mn.$$

The factor m^2+1 in the hypotenuse $(m^2+1)n$ is always positive; therefore, in order that the hypotenuse may be positive, n must be positive; then, in order that the side $2mn$ may be positive, m must be positive; and in order that the side $(m^2-1)n$ may be positive, m must be >1. The species of the triangle depends solely upon the value of m; if m is kept constant, and n varied, a series of similar triangles will be obtained.

It is often more convenient to take $\dfrac{1}{n}$ instead of n, and thus the sides would be $\dfrac{m^2+1}{n}, \ \dfrac{m^2-1}{n}, \ \dfrac{2m}{n}$.

I have found the results of this article very useful in solving the problems of Diophantus' Sixth Book.

7. The following general problem includes Diophantus VI., 6, 7, 8, 9, 10, 11, and the similar problems which have been added by Bachet and Fermat.*

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* Cf. Heath's *Diophantus*, pp. 227–229.
Bachet's *Diophantus*, pp. 383–388.
Oeuvres de Fermat, t. I., pp 329–331.

To find a rational right-angled triangle such that the sum of given multiples of the area and of the three sides may be equal to a given number.

Let a, b, c, d be the given multiples, e the given number, and $\frac{x^2+1}{y}$, $\frac{x^2-1}{y}$, $\frac{2x}{y}$ the sides of the required triangle;

then $$a\,.\,\frac{(x^2-1)x}{y^2}+b\,.\,\frac{x^2+1}{y}+c\,.\,\frac{x^2-1}{y}+d\,.\,\frac{2x}{y}=e;$$

$$\therefore\quad ey^2-\{(b+c)x^2+2dx+(b-c)\}y-a(x^2-1)x=0.$$

In order that this quadratic equation may give a rational value for y, we must have

$$\{(b+c)x^2+2dx+(b-c)\}^2+4ae(x^2-1)x=\text{a square},$$

which can in general be resolved for x in six ways, since the term in x^4 and the constant are both squares.*

The corresponding values for y can then be found from the quadratic equation. If x' is any one of the six values found for x, by putting $x=X+x'$, six new values may be obtained for x, and so on.

Hence the problem admits of an infinite number of solutions.

8. An example of the use of the method of § 4 and of § 6 is furnished by the following problem, which is necessary for the completion of the solution of Diophantus V., 24 and 25.†

To find two rational right-angled triangles such that their areas may be in a given ratio.

Let the given ratio be $r:s$,

the sides of the first triangle $m(x^2+1)$, $m(x^2-1)$, $2mx$,

and those of the second $n(y^2+1)$, $n(y^2-1)$, $2ny$.

Then $$\frac{m^2(x^2-1)x}{n^2(y^2-1)y}=\frac{r}{s};$$

$$\therefore\quad m^2s(x^2-1)x-n^2r(y^2-1)y=0.$$

Suppose the equation to be represented by a graph;

when $x=-1$ or 0 or 1 then $y=-1$ or 0 or 1.

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* Cf. Euler's *Algebra*, English translation, Part II., chap. IX., § 134.

† Cf. *Oeuvres de Fermat*, t. I., pp. 318–325.

Thus

$(-1,-1), (-1,0), (-1,1), (0,-1), (0,0), (0,1), (1,-1), (1,0), (1,1)$

are points on the graph.

The straight lines joining twelve pairs of these points will cut the graph again in a third point, which is different from any of the above points, and which necessarily has rational coordinates.

(i) The straight line joining $(0, -1)$ to $(1, 0)$ is $y=x-1$, and it cuts the graph where

$$m^2s(x^2-1)x - n^2r\{(x-1)^2-1\}(x-1)=0;$$

$$\text{i.e.,}\quad (n^2r-m^2s)x^3 - 3n^2rx^2 + (2n^2r+m^2s)x = 0;$$

$$\therefore\quad x=0 \text{ or } 1 \text{ or } \frac{2n^2r+m^2s}{n^2r-m^2s};$$

$$\therefore\quad x=\frac{2n^2r+m^2s}{n^2r-m^2s},\quad y=\frac{n^2r+2m^2s}{n^2r-m^2s},$$

where m and n can be taken arbitrarily.

e.g., if $m=1$, $n=1$, $x=\dfrac{2r+s}{r-s}$, $y=\dfrac{r+2s}{r-s}$.

Hence the sides of the first triangle are

$$\left\{\frac{(2r+s)^2}{(r-s)^2}+1\right\},\quad \left\{\frac{(2r+s)^2}{(r-s)^2}-1\right\},\quad \frac{2(2r+s)}{r-s}$$

and those of the second

$$\left\{\frac{(r+2s)^2}{(r-s)^2}+1\right\},\quad \left\{\frac{(r+2s)^2}{(r-s)^2}-1\right\},\quad \frac{2(r+2s)}{r-s};$$

or, multiplying both sets by $(r-s)^2$, we may take the sides of the first triangle to be

$$\{(2r+s)^2+(r-s)^2\},\quad \{(2r+s)^2-(r-s)^2\},\quad 2(2r+s)(r-s)$$

and those of the second

$$\{(r+2s)^2+(r-s)^2\},\quad \{(r+2s)^2-(r-s)^2\},\quad 2(r+2s)(r-s);$$

that is, the first triangle may be formed from $(2r+s, r-s)$ and the second from $(r+2s, r-s)$. This is Fermat's first method, which he merely states, without showing how it was obtained.*

It is evident that it is only one of an infinite group of methods.

* Cf. *Oeuvres de Fermat*, t. I., p. 320.

If $r=3$, $s=1$, the first triangle, formed from (7, 2), is (53, 45, 28)
and the second, formed from (5, 2), is (29, 21, 20).

(ii) The straight line joining (0, 1) to (1, 0) cuts the graph again at

$$x=\frac{2n^2r-m^2s}{n^2r+m^2s}, \quad y=\frac{2m^2s-n^2r}{n^2r+m^2s},$$

where m and n are arbitrary.

If $m=1$, $n=1$, $$x=\frac{2r-s}{r+s}, \quad y=\frac{2s-r}{r+s}.$$

Thus the first triangle may be formed from $(2r-s, r+s)$ and the second from $(r+s, r-2s)$, which is Fermat's second method.*

If $r=3$, $s=1$, the first triangle, formed from (5, 4), is (41, 9, 40)
and the second, formed from (4, 1), is (17, 15, 8).

(iii) The straight line joining $(-1, -1)$ to (1, 0) cuts the graph again at

$$x=\frac{3n^2r}{n^2r-8m^2s}, \quad y=\frac{n^2r+4m^2s}{n^2r-8m^2s},$$

where m and n are arbitrary.

If $m=1$, $n=4$, $$x=\frac{6r}{2r-s}, \quad y=\frac{4r+s}{4r-2s}.$$

Hence the first triangle may be formed from $(6r, 2r-s)$ and the second from $(4r+s, 4r-2s)$. This is Fermat's third method.*

If $r=3$, $s=1$, the first triangle, formed from (18, 5), is (349, 299, 180)
and the second, formed from (13, 10), is (269, 69, 260).

(iv) The straight line joining $(-1, 1)$ to (1, 0) cuts the graph again at

$$x=\frac{3n^2r}{n^2r+8m^2s}, \quad y=\frac{4m^2s-n^2r}{n^2r+8m^2s},$$

where m and n are arbitrary.

If $m=1$, $n=4$, $$x=\frac{6r}{2r+s}, \quad y=\frac{s-4r}{4r+2s}.$$

* Cf. *Oeuvres de Fermat*, t. I., p. 320.

Thus the first triangle may be formed from $(6r,\ 2r+s)$ and the second from $(4r+2s,\ 4r-s)$, which is Fermat's fourth method.*

If $r=3$, $s=1$, the first triangle, formed from (18, 7), is (373, 275, 252) and the second, formed from (14, 11), is (317, 75, 308).

(v) The straight line joining $(0,\ -1)$ to $(1,\ 1)$ cuts the graph again at

$$x=\frac{4n^2r+m^2s}{8n^2r-m^2s},\quad y=\frac{3m^2s}{8n^2r-m^2s},$$

where m and n are arbitrary.

If $m=4$, $n=1$, $\qquad x=\dfrac{r+4s}{2r-4s},\quad y=\dfrac{6s}{r-2s}$.

Therefore the first triangle may be formed from $(r+4s,\ 2r-4s)$ and the second from $(6s,\ r-2s)$.

If $r=3$, $s=1$, the first triangle, formed from (7, 2), is (53, 45, 28) and the second, formed from (6, 1), is (37, 35, 12).

(vi) The straight line joining $(0,\ 1)$ to $(1,\ -1)$ cuts the graph again at

$$x=\frac{4n^2r-m^2s}{8n^2r+m^2s},\quad y=\frac{3m^2s}{8n^2r+m^2s},$$

where m and n are arbitrary.

If $m=4$, $n=1$, $\qquad x=\dfrac{r-4s}{2r+4s},\quad y=\dfrac{6s}{r+2s}$.

Thus the first triangle may be formed from $(2r+4s,\ 4s-r)$ and the second from $(6s,\ r+2s)$.

If $r=3$, $s=1$, the first triangle, formed from (10, 1), is (101, 99, 20) and the second, formed from (6, 5), is (61, 11, 60).

Fermat does not give a particular case of v and vi.

No results essentially different from the above are obtained by joining the remaining six pairs of points.

Since m and n can be taken arbitrarily in any of the above, it is clear that the two triangles can be formed in an infinite number of ways. Also, starting from any of the above solutions we could deduce an infinite number of others by the tangent-method; but these solutions would be of a much more complex nature.

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* *Oeuvres de Fermat*, t. I., p. 320.

Note on the Linear Matrix Equation.

By J. H. Maclagan-Wedderburn, M.A.

The general linear matrix equation has been investigated by Sylvester* in a series of interesting papers published in the *Comptes Rendus.* His method, however, does not show that the solution can be exhibited as an analytical function of the coefficients. The object of this note is to supply such a solution.

The term, analytic function of the matrices $\phi_1, \dots \phi_n$, is used in this paper to denote any function of $\phi_1, \dots \phi_n$, which can be developed in a convergent series involving only powers of the variables and constants, *i.e.*, a series of the form

$$f(\phi) = \theta_1 + \Sigma\theta_2\phi\theta_3 + \Sigma\theta_4\phi\theta_5\phi\theta_6 + \dots$$

where the θ's are matrices independent of ϕ.

The simplest case of the linear matrix equation is

$$\chi - \phi\chi\psi = \theta. \qquad (1)$$

This gives

$$\chi = \theta + \phi\chi\psi$$

which, when substituted in the second term of the left-hand side of (1), leads to

$$\chi - \phi^2\chi\psi^2 = \theta + \phi\theta\psi$$

and, on repeating the process n times,

$$\chi - \phi^{n+1}\chi\psi^{n+1} = \theta + \phi\theta\psi + \phi^2\theta\psi^2 + \dots + \phi^n\theta\psi^n$$
$$= \theta_n \qquad (2)$$

From this it is evident that the series $\underset{n=\infty}{\mathrm{L}}\ \theta_n$, when convergent, is a solution of (1).

* *Comptes Rendus*, I.C., 1884, p. 67, 115, 117, 527.

The general linear equation

$$\Sigma\phi\chi\psi = \theta,$$

when arranged in the form

$$\chi - \Sigma\phi_1\chi\psi_1 = \theta,$$

can be treated in the same way, the solution being

$$\chi = \theta + \Sigma\phi_s\theta\psi_s + \ldots\ldots + \Sigma\phi_{s_1}\phi_{s_2}\ldots\phi_{s_n}\theta\psi_{s_n}\ldots\psi_{s_2}\psi_{s_1} + \ldots$$

This can be put in the simple form

$$\chi = \theta + \Phi\theta + \Phi^2\theta + \ldots \quad - \quad - \quad - \quad - \quad (3)$$

where Φ denotes the operation* $\Sigma\phi_s(\ )\psi_s$

i.e., Φ is a matrix of order n^2 where n is the order of the matrices θ, χ, etc.

The series

$$1 + \Phi + \Phi^2 + \ldots\ldots$$

evidently converges if

$$1 + g + g^2 + \ldots\ldots$$

converges for all the roots of Φ. (See E. Weyr, *Bull. des Sc. Math.*, 11 (1887), p. 205.) It is obvious that, by manipulating the original equation, we can in general secure the convergence of these series.

This method of solution does not depend on the nature of the symbols used, and can be applied in a variety of problems.

We may use it, for instance, to solve the equation

$$\frac{\partial\chi}{\partial t_1} + \Sigma\phi_r\frac{\partial\chi}{\partial t_r}\psi_r = \frac{\partial\chi}{\partial t_1} - U\chi = \theta. \quad - \quad - \quad - \quad - \quad (4)$$

The inverse of $\frac{\partial}{\partial t_1}$ is $\int_{a_1}^{t} dt_1$, a_1 being an arbitrary function of $t_2 \ldots t_n$,

and the solution of (4) is accordingly

$$\chi = \int_{a_1}^{t_1} dt_1\theta + \int_{a_2}^{t_1} dt_1 U \int_{a_1}^{t_1} dt_1\theta + \int_{a_3}^{t_1} dt_1 U \int_{a_1}^{t_1} dt_1 U \int_{a_2}^{t_1} dt_1\theta + \ldots\ldots$$

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* Sylvester *Comptes Rendus*, 1884, p. 117.

This method is capable of considerable extension, *e.g.*, the solution of

$$\frac{\partial^2 f}{\partial x^2}+\xi(x, y, z)\frac{\partial^2 f}{\partial y^2}+\eta(x, y, z)\frac{\partial^2 f}{\partial z^2}\equiv\frac{\partial^2 f}{\partial x^2}+\mathrm{U}f=0$$

is * $$f=\left(1-\int_{x_2}^{x}dx\int_{x_1}^{x}dx\mathrm{U}+\int_{x_4}^{x}dx\int_{x_3}^{x}dx\mathrm{U}\int_{x_2}^{x}dx\int_{x_1}^{x}dx\mathrm{U}\ldots\ldots\right)g(y, z)$$

$$+\left((x-x_0)-\int_{x_2}^{x}dx\int_{x_1}^{x}dx\mathrm{U}(x-x_0)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\right)h(y, z)$$

where $x_0, x_1, \ldots g(y, z)$ and $h(y, z)$ are arbitrary functions of y and z. If all the lower limits of integration are equal to x_0, the solution is such that, when $x=x_0$, then $f=g(y, z)$ and $\frac{\partial f}{\partial x}=h(y, z)$. The solution therefore satisfies Cauchy's initial conditions. The convergency of such series is often easily investigated. They also give a variety of ways of expanding functions in series whose convergence is readily tested. All the quantities used above may be matrices except the independent variables and the limits of integration.

The following method of solving the equation

$$\phi\chi-\chi\psi=\theta \qquad (5)$$

is a direct generalisation of the method which Hamilton used for the corresponding equation in quaternions. Its chief interest lies in showing the connection between the methods of Hamilton and Sylvester. Taber † has defined a series of matrices denoted by $\mathrm{K}\phi, \mathrm{K}^2\phi\ldots\mathrm{K}^{n-1}\phi$ which are obtained from ϕ by $1, 2, \ldots n-1$ cyclical permutations of its roots, n being the order of ϕ. He has also shown that the following relations subsist between them :

* Sturm (*Cours d'Analyse* II., p. 146, § 614) has obtained the same series by a somewhat similar method for the equation $\frac{d^2y}{dx^2}+\mathrm{I}y=0$. He, however, makes all his limits of integration the same. This series has also been obtained by different methods by other authors, *e.g.*, Caqué, *Liouville's Journal*, Ser. II., Vol. IX., 1864, and Fuch's *Annali di Math.*, Ser. II., Vol. 4, p. 46. See also *Trans. Roy. Soc. Edin.*, XL., 1903. This method of solution can evidently be extended to equations of any order.

† *Amer. Journ. of Math.*, Vol. XII. (1890), p. 388.

$$\left.\begin{aligned}
&\phi + K\phi + K^2\phi + \ldots\ldots + K^{n-1}\phi = -m_1 \\
&\phi K\phi + K\phi K^2\phi + \ldots\ldots + K^{n-1}\phi\phi = m_2 \\
&\ldots\ldots\ldots\ldots\ldots \\
&\Sigma\phi K\phi K^2\phi\ldots\ldots K^r\phi \qquad = (-1)^{r+1}m_{r+1} \\
&\ldots\ldots\ldots\ldots\ldots \\
&\phi K\phi K^2\phi\ldots\ldots K^{n-1}\phi \qquad = (-1)^n m_n
\end{aligned}\right\} \quad (6)$$

where $m_1, m_2, \ldots$ are the coefficients in the characteristic equation of ϕ, viz.,

$$\phi^n + m_1\phi^{n-1} + \ldots\ldots + m_n = 0. \quad (7)$$

We can now derive from (5) a series of equations of the form

$$K^{r_1}\phi K^{r_2}\phi\ldots K^{r_{n-1}}\phi\phi\chi\psi^s - K^{r_1}\phi\ldots K^{r_{n-1}}\phi\chi\psi^{s+1}$$
$$= K^{r_1}\phi\ldots K^{r_{n-1}}\phi\theta\psi^s,$$

the r's being given all values from 0 to $n-1$ with the following restrictions: (1) expressions differing only in the order of their terms are to be regarded as identical; (2) in one expression none of the r's are to be equal; (3) $s = n - 1 - r_1 - r_2\ldots - r_{n-1} \nless 0$.

Adding all these equations together we get the following result:

$$K^{n-1}\phi K^{n-2}\phi\ldots K\phi\phi\chi - (\Sigma K^{n-2}\phi K^{n-1}\phi\ldots K\phi\phi)\chi\psi + \ldots\ldots$$
$$\pm(\phi + K\phi + K^2\phi + \ldots + K^{n-1}\phi) + (-1)^n\chi\psi^n = K^{n-1}\phi K^{n-2}\phi\ldots K\phi\theta + \ldots$$
$$= \Theta$$

i.e., $\chi(m_n + m_{n-1}\psi + \ldots + \psi^n) = -\Theta.$

$\therefore \quad \chi = -\Theta(m_n + m_{n-1}\psi + \ldots + \psi^n)^{-1}.$

This is equivalent to Sylvester's form in which Θ is obtained as

$$\begin{aligned}
&(\phi^{n-1}\theta + \phi^{n-2}\theta\psi + \ldots + \theta\psi^{n-1}) \\
&+ m_1(\phi^{n-2}\theta + \ldots + \theta\psi^{n-2}) \\
&+ \ldots\ldots\ldots\ldots\ldots \\
&+ m_{n-1}(\phi\theta + \theta\psi).
\end{aligned}$$

A curious consequence of this result is that if ψ has the same characteristic equation as ϕ, then

$$\theta = \phi\chi - \chi\psi$$

must satisfy the linear equation

$$\Theta = 0.$$

If $\phi = \psi$ we can also find as follows a set of scalar conditions satisfied by the coefficients of θ.

Substituting $\phi + k\theta$ for ϕ in (7) and equating the coefficient of k to zero, we get the following equation *

$$\Theta + a_1\phi^{n-1} + a_2\phi^{n-2} + \ldots + a_n = 0,$$

where $a_1, \ldots a_n$ are invariants of the two matrices ϕ and θ; therefore, as $\Theta = 0$, we must also have

$$a_1 = 0, \;\; a_2 = 0, \;\; \ldots\ldots \;\; a_n = 0.$$

Similar conclusions can be drawn with regard to the invariants of χ and θ.

* Cf. Sylvester, *Amer. Journ. of Math.*, VI., 1884, p. 279.

On the Use of Symmetry in Geometry.

By JOHN W. BUTTERS, M.A., B.Sc.

Two features characterise the treatment of Geometry as presented in Euclid's "Elements": (1) the propositions are arranged in a definite sequence which cannot be greatly altered without invalidating the proofs; (2) there are no methods of proof applicable to a large number of propositions. If we except the method of *reductio ad absurdum*, it is scarcely an exaggeration to say, for example, that in Book I no three propositions are proved by the same method.

Now that the order of Euclid has been abandoned it is desirable (1) that the proofs should be made as far as possible independent of the new order adopted, and (2) that methods having a wide application should be preferred to those suitable only for the proposition on hand.

It is the object of this paper to show that the general use of *symmetry* (in its various forms) would be of advantage.

The subject may be approached experimentally. Fold a sheet of paper along a line drawn on it and prick a number of holes through the double paper. Now unfold the paper. The holes (points) occur in pairs, one on each side of the line of fold (the axis of symmetry). The two points forming a pair are called corresponding points and, in general, lines, angles, triangles, etc., which coincide when the paper is folded (and are therefore congruent) are called corresponding lines, angles, triangles, etc.

The following propositions are obvious. (Fig. 6.)

1. Each point on the axis corresponds to itself, and, conversely, if a point corresponds to itself it is on the axis.

2. The axis corresponds to itself, as does every line perpendicular to the axis, and, conversely, if a line corresponds to itself it is an

axis of symmetry or is perpendicular to an axis of symmetry. (*Note*: a figure may have more than one axis of symmetry.)

3. The line passing through two points corresponds to the line passing through the corresponding points, and, conversely, if two lines correspond they pass through corresponding points.

4. The distance between two points is equal to the distance between the corresponding points; conversely, if two points on two corresponding lines correspond, then if equal distances be measured from these points in the corresponding directions (both towards or both away from the axis) along these lines, the points thus found are corresponding points.

5. The angle made by a pair of lines is equal to the corresponding angle made by the corresponding lines, and conversely. (*Note*: if we consider the direction of a line towards the axis to be positive in all cases, then corresponding angles are of opposite sense.)

6. The distance of a line from a point is equal to the distance of the corresponding line from the corresponding point. (This assumes that only one perpendicular can be drawn from a point to a line.)

Many more similar propositions may be enunciated but the above are sufficient to show that the method is applicable to a large number of Euclid's propositions.

Examples.

I. 5. Given $AB = AC$, to prove that $\widehat{B} = \widehat{C}$. (Fig. 7.)

Draw AP, the bisector of $\widehat{A}$, as axis; this line may be called simply the axis of $\widehat{A}$.

Since AP corresponds to AP (Prop. 2.) and $P\widehat{A}B = P\widehat{A}C$,

$\therefore$ AB corresponds to AC. (Prop. 5 Converse.)

Since A corresponds to A and $AB = AC$,

$\therefore$ B corresponds to C; (Prop. 4 Converse.)

$\therefore$ BC corresponds to CB; (Prop. 3.)

$\therefore$ $\widehat{B} = \widehat{C}$. (Prop. 5.)

I. 6. Given $\widehat{B} = \widehat{C}$, to prove that BA = CA. (Fig. 8.)

Draw PQ, the perpendicular bisector of BC; this line may be called the axis of BC.

P corresponds to P and PB = PC;

$\therefore$ B corresponds to C. (Prop. 4 Converse.)

B corresponds to C and $\widehat{B} = \widehat{C}$;

$\therefore$ BA corresponds to CA; (Prop. 5 Converse.)

$\therefore$ A corresponds to A.

B corresponds to C, A corresponds to A:

$\therefore$ BA = CA. (Prop. 4.)

I. 18. Give AC > AB, to prove $\widehat{B} > \widehat{C}$. (Fig. 9.)

Let the axis of $\widehat{A}$ meet BC in D; then B′, the point corresponding to B, will lie in AC, the line corresponding to AB.

Since AC > AB, B′ will lie between A and C;

hence $A\widehat{B'}D > \widehat{C}$;

but $\widehat{B} = A\widehat{B'}D$, these being corresponding angles;

$\therefore \widehat{B} > \widehat{C}$.

I. 19. Given $\widehat{B} > \widehat{C}$, to prove CA > BA. (Fig. 9.)

Let the axis of $\widehat{A}$ meet BC in D;

since $B\widehat{A}D + \widehat{B} + A\widehat{D}B = D\widehat{A}C + \widehat{C} + A\widehat{D}C$

and $B\widehat{A}D = D\widehat{A}C$

also $\widehat{B} > \widehat{C}$;

$\therefore A\widehat{D}B < A\widehat{D}C$.

Let DB′ correspond to DB,

then $A\widehat{D}B' = A\widehat{D}B$ and $\therefore A\widehat{D}B' < A\widehat{D}C$;

$\therefore$ B′ lies between A and C;

$\therefore$ AB′ < AC and $\therefore$ AB < AC.

I. 20. In the figure to the last proposition join BB';

then $D\widehat{B'}B$ corresponds to $D\widehat{B}B'$;

$$\therefore \quad D\widehat{B'}B = D\widehat{B}B';$$

$$\text{hence} \quad C\widehat{B'}B > C\widehat{B}B';$$

$$\therefore \quad BC > B'C \;;$$

$$\text{but} \quad AB = AB' \;;$$

$$\therefore \quad AB + BC > AC.$$

(This proof supposes that $AB < AC$; if $AB \nless AC$, no proof is needed.)

Note: Since $BC > B'C$ and B, B′ are corresponding points, we get the theorem: if lines be drawn from a point, not on the axis, to corresponding points, the one cutting the axis is greater than the other, and conversely.

I. 24. (Fig. 10.)

Let the triangles be ABC, ABD, having one pair of equal sides coinciding, and let $B\widehat{A}D$ be less than $B\widehat{A}C$. Then the axis of $D\widehat{A}C$ will fall within the larger angle BAC and will $\therefore$ cut BC; but C and D are corresponding points; $\therefore$ $CB > DB$. (Theorem just proved.)

I. 25 may be proved as above, using the converse of the theorem.

Otherwise: Draw PQ the axis of DC (Fig. 10.); then

$$\text{since} \quad A\widehat{D}C = A\widehat{C}D,$$

$\therefore$ DA corresponds to CA;

$\therefore$ A lies on PQ.

Now PQ cuts the greater line BC;

$$\therefore \quad C\widehat{A}B > D\widehat{A}B.$$

These theorems have all been proved independently of the Congruence Theorems I. 4, 8, 26. These may be proved by the same method.

I. 4. (Fig. 11.)

Place the triangles so that they have a pair of equal sides AB

coincident and so that the other equal sides AC, AC′ lie on opposite sides of AB; then $B\widehat{A}C$ will be equal to $B\widehat{A}C'$.

AB is an axis of symmetry of AC, AC′, and since AC = AC′,

∴ C corresponds to C′ and hence BC corresponds to BC′;

∴ BC = BC′; also $A\widehat{B}C = A\widehat{B}C'$ and $\widehat{C} = \widehat{C'}$.

I. 8. (Fig. 11.)

In the triangles ABC, ABC′, AB is common, AC = AC′ and BC = BC′.

C, C′ lie on the circumference of a circle whose centre is A and which therefore has AB as an axis of symmetry.

C, C′ lie also on the circumference of a circle whose centre is B and which therefore also has AB as an axis of symmetry.

Hence AB is an axis of symmetry of the whole figure.

Now the point corresponding to C must lie on the circle A, since C lies on that circle; similarly it must lie on circle B; hence, since these circles have only one other point in common (viz. C′), C corresponds to C′;

∴ CA corresponds to C′A and BC corresponds to BC′;

∴ $C\widehat{A}B = C'\widehat{A}B$, $C\widehat{B}A = C'\widehat{B}A$ and $\widehat{C} = \widehat{C'}$.

I. 26. (Fig. 11.)

Place the equal sides of the triangles so that they coincide and take this line AB as axis.

Given $C\widehat{A}B = C'\widehat{A}B$; ∴ CA corresponds to C′A;

also $A\widehat{B}C = A\widehat{B}C'$; ∴ BC corresponds to BC′;

∴ C corresponds to C′;

hence CA = C′A, BC = BC′ and $\widehat{C} = \widehat{C'}$.

When the given equal angles are CAB, C′AB and C, C′ we have $C\widehat{B}A = C'B\widehat{A}$ by I. 32 which may be proved without assuming I. 26.

For convenience, the propositions depending on *Axial Symmetry* have been grouped together, although this has necessitated the assumption of the truth of I. 16, I. 32.

Let us now consider *Central Symmetry.*

Suppose the line-segment AB (Fig. 12.) to rotate through two right angles about its middle point C, then CA and CB would change places, A would take the place of B and B the place of A. Further, if $C\widehat{A}D = C\widehat{B}E$, then AD and BE would interchange places. Also, if AD = BE then D and E would interchange places; and so on.

A figure which after a rotation through two right angles occupies the same position is said to have Central Symmetry, the centre of rotation being the centre of symmetry; parts which interchange positions are called corresponding parts.

The following propositions are obvious; they should be compared with the corresponding ones in Axial Symmetry. (Fig. 13.)

1. Each line passing through the centre corresponds to itself, and, conversely, if a line corresponds to itself it passes through the centre.

2. The centre corresponds to itself, and, conversely, if a point corresponds to itself it is the centre of symmetry.

3. The point of intersection of two lines corresponds to the point of intersection of the corresponding lines, and, conversely, if two points correspond, corresponding lines pass through them.

4. The angle between two lines is equal to the angle between the corresponding lines, and, conversely, if two lines correspond and at corresponding points on them equal angles are formed in the same direction (both clockwise or both counter-clockwise), then the other arms of these angles are corresponding lines.

5. The distance between two points is equal to the distance between the corresponding points, and conversely.

(*Note*: It will be shown that corresponding lines are parallel; corresponding distances are therefore of opposite sense.)

It will be noticed that for every statement made in axial symmetry there is a corresponding statement in central symmetry derived from it by the interchange of point for line and line for point. We thus get early an introduction to the principle of Duality.

Examples of Central Symmetry.

I. 15. (Fig. 14.)

E is a centre of symmetry;
EB corresponds to EA;
ED corresponds to EC;
$\therefore \quad B\widehat{E}D = A\widehat{E}C$.
(Prove $B\widehat{E}C = A\widehat{E}D$ similarly).

I. 27. (Fig. 15.)

Given $A\widehat{G}H = D\widehat{H}G$.
Take O the mid-point of GH as centre.
OG corresponds to OH; (Prop. 1.)
and since OG = OH,
$\therefore$ G corresponds to H; (Prop. 5 Converse.)
Because $O\widehat{G}A = O\widehat{H}D$,
$\therefore$ GA corresponds to HD, (Prop. 4.)
and consequently HC corresponds to GB.

Hence by Prop. 3 the point of intersection of GA and HC (if any) corresponds to the point of intersection of HD and GB (if any); *i.e.*, if there is *one* point of intersection of AB and CD, then there are *two*. As this is impossible, AB and CD have no point of intersection, *i.e.*, they are parallel, *i.e.*, corresponding lines are parallel.

I. 29. (Fig. 15.)

Given AB parallel to CD.
Take O the mid-point of GH as centre of symmetry.
Then H corresponds to G;
$\therefore$ the line corresponding to CHD passes through G;
it is also parallel to CD (I. 27) and since there is only one parallel to CD through G (viz., AB),
$\therefore$ AB corresponds to CD;
$\therefore \quad A\widehat{G}H = G\widehat{H}D$.

Corollary.—If two lines are parallel and pass through corresponding points, the lines correspond.

I. 33. (Fig. 16.)

Given AB equal and parallel to CD.

Take the mid-point of CB as centre of symmetry.

B corresponds to C, and BA is parallel to CD;

$\therefore$ BA corresponds to CD.

Also BA = CD;

$\therefore$ A corresponds to D;

but C corresponds to B;

$\therefore$ AC corresponds to DB;

$\therefore$ AC = BD and AC is parallel to BD.

I. 34. (Fig. 16.)

Given AB parallel to CD and AC parallel to BD.

Take the same centre of symmetry as in I. 33.

Because B corresponds to C and BA is parallel to CD,

$\therefore$ BA corresponds to CD.

Again C corresponds to B and CA is parallel to DB,

$\therefore$ CA corresponds to DB.

Hence A, the intersection of BA, CA corresponds to D, the intersection of CD, BD;

$$\therefore \quad AB = CD, \quad AC = BD, \quad \widehat{A} = \widehat{D}, \text{ etc.}$$

Examples of application to problems.

I. 9. (Fig. 17.)

Take D, D′ and also E, E′ pairs of corresponding points on BA, BC. Let the corresponding lines ED′, E′D meet in F. Then F corresponds to itself; hence F lies on the axis; BF is the axis; $\therefore$ $F\widehat{B}E = F\widehat{B}E'$.

I. 10. (Fig. 18.)

The above suggests the corresponding solution in central symmetry, viz.:—take *d*, *d′* and also *e*, *e′* pairs of corresponding lines through R and S. Let the join of the points *ed′*, *e′d* be *f*. Then *f* corresponds to itself and therefore passes through the centre of symmetry, say O; hence OR = OS.

In central symmetry the lines joining corresponding points pass through the centre and the points are equidistant from the centre. If the distances instead of being equal have a constant ratio, we get the theory of similar figures, which is beyond the scope of this paper.

In axial symmetry the lines joining corresponding points make equal angles with the axis and the points are equidistant from the axis. If the angles instead of being equal have a constant ratio (*i.e.*, if the lines instead of being perpendicular to the axis make a constant angle with it), we get the theory of skew-symmetrical figures.

Let A, A′; B, B′ (Fig. 19.) be pairs of corresponding points with reference to the skew axis PQ; *i.e.*, BP = PB′, AQ = QA′ and AA′ is parallel to BB′.

Let the figure A′B′PQ be rotated about O, the mid point of PQ, through two right angles. Then QA′ will lie along PB as PA″; also PB′ will lie along QA as QB″.

Since PB = QB″ and PA″ = QA,

∴ BB″ and A″A are parallel to PQ.

Hence ABB″ = A″B″B, each being one half of the parallelogram AA″BB″;

to these equals add BQ and we have

ABPQ = A″B″QP = A′B′PQ.

Consider now two corresponding triangles ABC, A′B′C′ (Fig. 20.), and let AA′, BB′, CC′ cut the axis in P, Q, R.

Then the area of ABC is equal to the algebraic sum of the trapezia ABQP, BCRQ, CAPR; and the area of A′B′C′ equals the sum of the trapezia A′B′QP, B′C′RQ, C′A′PR, those areas being reckoned positive when they lie to the left hand in going round in the order of the letters.

As the corresponding trapezia are equal but of opposite sign, the same is true of the triangles and hence of figures in general. Hence, in skew symmetry, corresponding areas are equal.

Euclid's theorems in areas follow at once.

Examples.

I. 36. (Fig. 21.)

Take as skew axis the line joining the mid points of DE and CF; then ABCD, HGFE are corresponding areas and are ∴ equal.

I. 43. (Fig. 22.)

EG, BD, HF being diagonals of parallelograms, are bisected by AKC and these lines are parallel; (Proof?)

∴ AC is a skew axis and EBHK corresponds to GDFK;

∴ EBHK = GDFK.

(The foregoing includes all the main propositions of Euclid's first book except I. 47 and its converse).

As another example take AD, the median to BC as a skew axis (Fig. 23);

then AB corresponds to AC;

F the mid point of AB corresponds to E the mid point of AC;

∴ BE corresponds to CF;

hence they intersect on AD; *i.e.*, the medians are concurrent.

It is not the purpose of this paper to apply the method of symmetry beyond the subject matter of Euclid's first book. It is obvious, however, that it is of ready application to the properties of circles as developed in the third and fourth books.

In these days of the use of squared paper the transformation of a circle into an ellipse may form an appropriate conclusion:—

Draw a circle on squared paper and a line not perpendicular to the lines which cross the paper horizontally; take this line as skew axis (Fig. 24.).

Corresponding to each point of intersection of the circle with the series of horizontal lines, a point is taken on the opposite side of the skew axis and equidistant from it. By joining the series of points thus obtained we get an ellipse, and by counting the squares enclosed by each curve, the equality of the areas may be tested. Other properties may also be readily obtained; *e.g.*, since the mid-points of the horizontal chords of the circle transform into the mid-points of the corresponding chords of the ellipse, the locus of these is a straight line.

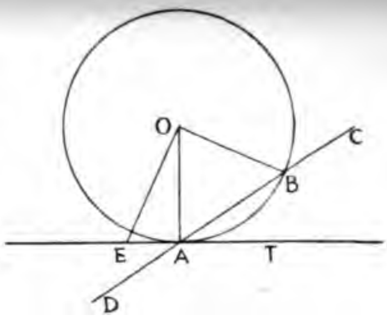
Sixth Meeting, 13th May 1904.

Mr CHARLES TWEEDIE, President, in the Chair.

Note on the Treatment of Tangents in recent Textbooks of Elementary Geometry.

By Professor GEORGE A. GIBSON.

Several textbooks of Elementary Geometry have recently been put on the market, and in nearly all that I have examined (and I have gone carefully through many of them) the treatment of tangents is based on what the writers call the Method of Limits. The usual form given to the proof that the tangent at any point of a circle is at right angles to the radius to the point of contact is somewhat as follows.



The radii OA, OB are equal and therefore

$$\angle OBA = \angle OAB \quad (1)$$

This equation is true however near B is to A; when B coincides with A the angle AOB is zero, and the angle OAB is a right angle. But when B coincides with A the secant ABC is the tangent, and therefore the tangent at A is at right angles to OA.

Sometimes, instead of the angles OAB, OBA the supplements OAD, OBC are taken and instead of (1) we have

$$\angle OAD = \angle OBC \quad - \quad - \quad - \quad - \quad (2)$$

but this makes no real difference in the proof; neither C nor D can be found except through the *two* points A and B.

It is more important, however, to note that some writers explicitly state an assumption which all who adopt this mode of

proof actually make, either explicitly or implicitly: namely, after the words "this equation is true however near B is to A" they add "therefore it is true when B coincides with A." This assumption is of course identical with that implied in the venerable dictum that "what is true up to the limit is true in the limit."

Now, it is surely not hypercritical to call in question the logic of this proof. So far as the reasoning is concerned, in what respect does it differ from the following? On the line through A at right angles to OA take any point E distinct from A. The angle OAE is greater than the angle OEA and therefore

$$OE > OA \qquad (3)$$

This inequality is true however near E is to A and therefore it is true when E coincides with A; that is, OA is greater than itself.

As a mere matter of reasoning, the conclusion is as sound in the one case as in the other, on the assumption of the dictum quoted.

Of course the whole difficulty lies in the failure to grasp, or at least to state and apply, the proper definition of a limit. It is rather disheartening to find the absurdities, so clearly pointed out by Berkeley nearly two hundred years ago, still flourishing and apparently endowed with a new lease of life. It is all the more regrettable to find these in English textbooks, when one considers that we owe to one Englishman the explicit statement, and to another a thoroughly satisfactory exposition, of the Method of Limits. (See the notice of the *Analyst* Controversy, *Proc. Edin. Math. Soc.*, Vol. XVII., pages 9–32).

The radical error of all such proofs as that sketched above lies, it seems to me, in a wrong conception of a limit; a limit seems to be considered *as a particular case.* Thus the straight line OA (or the two *coincident* straight lines OA, OB which still make but one line) is considered to be a particular case of the triangle OAB. But, however *convenient* it may be to use the language of coincident points and lines, there is absolutely no cogency in the reasoning that is often based on the conception of coincident points and lines. Equation (1) above is established on the express understanding that A and B are distinct points and cannot be established unless they are distinct; equation (1) (like the inequality (3)) is true only so long as B is distinct from A (or E distinct from A). When B coincides

with A, OAB is no longer a triangle. It is surely not going to be accepted as an axiom of the modern geometry that, when a theorem has been established on the express understanding that certain conditions hold, we are at liberty to maintain that the theorem is true when one or more of these conditions are violated. The theorem may be true when one or more of the conditions are violated, but that is a matter for proof and is not a legitimate assumption.

It is a mere commonplace of careful writers on mathematics that the limit to which a function $f(x)$ converges when x converges to a limit, a say, has by its definition nothing whatever to do with *the particular case* of the function when *x is equal to a*. In fact the reason for the introduction of the notion of a limit is, that the usual definition of the function ceases to give a definite meaning for the particular value a of x; though of course the definition of a limit holds equally well whether $f(a)$ has or has not a definite meaning when evaluated by the ordinary rules of algebra. It is not easy to say how much of the erroneous conception of a limit is due simply to defective language and notation; the phrase "when x is equal to a" in the clause "the limit of $f(x)$ when x is equal to a" has, I fear, led many astray. It cannot be too emphatically insisted upon that, in finding the limit of $f(x)$ when x converges to a as its limit, the value a must not be assigned to x; the limit depends, not on the value of $f(x)$ when x has the value a but on the values of $f(x)$ when x is *all but* equal to a. So far as the limit is concerned, it does not matter in the least whether $f(x)$ has or has not a definite value when x is equal to a; cases are quite common in which $f(x)$ has a definite value when x is equal to a and also a definite limit when x converges to a, and yet the value and the limit are not equal.

If the method of limits is to be used with absolute beginners in geometry (personally, I am inclined to hold that it is not suitable as a method of reasoning for absolute beginners) there should be greater care taken to show the *reasonableness* of the definition, and the proofs should be genuine and not merely plausible. For the beginner the process by which a secant through a fixed point outside a circle is gradually rotated till it becomes a tangent, is very valuable *by way of suggestion*, and a teacher who does not frequently use the process in order to gain theorems on tangents loses a great oppor-

tunity. But when the process has suggested a theorem, that theorem should be demonstrated by a method which implies that the tangent has been actually drawn. Thus, I think Euclid was wise in proving III 36 as well as III 35.

When the notion of a limit is first introduced, it should, I think, be strictly confined to the case of a tangent; the general definition is too abstract. I would suggest some such definition as the following :—the tangent at a point A on a curve is a line AT such that the angle TAB between AT and the secant AB, through A and any other point B on the curve near to A, is small when B is near to A, and can be made as small as we please simply by taking B near enough to A.

The definition is rather long-winded, but it merely states in other words what, I think, is the ordinary conception of a tangent; namely, the tangent at A is a line (i) that meets the curve in only one point near A but (ii) that, if rotated about A as a pivot through any small angle (no matter how small that angle may be) will again cut the curve near A.

Now, to prove that the tangent to a circle is the line at right angles to the radius to the point of contact, first draw AT perpendicular to OA. Then, since the angle TAB is half the angle AOB, that angle is small when B is near A and can obviously be made as small as we please by taking B near enough to A. Hence AT is the tangent at A.

I hesitated for some time about asking the Society to accept this Note, but I finally felt myself justified in making the request on considering that we are now at the beginning of a series of great changes in the teaching of mathematics, and that there is almost a consensus of opinion among recent writers of textbooks as to the treatment of tangents by the method of limits. The exposition actually given of the method seems to me to be so radically faulty and so well fitted to make it difficult for a pupil to gain a sound knowledge of the method in his later studies, that I have ventured to take up the time of the Society with matters that are certainly well understood and properly expounded in various works.

On the Use of the term *Power* in Geometry, and on the treatment of the "doubtful sign."

By R. F. MUIRHEAD.

Amongst the "technical terms" that have come into use in connection with Coordinate Geometry, not the least convenient is the word *Power*. The only definition of a general kind for this term that I have met with is the following:

"DEF.—*The result of substituting the coordinates of any point in the equation of any line or curve is called the* POWER *of that point with respect to the line or curve.*

"[This definition, first given by Steiner, is now employed by all the French and German writers.]"

This quotation is from Casey's *Treatise on Conic Sections*, page 26.

I have consulted Steiner's published works, and have found therein two definitions of the "Potenz" or Power of a point with respect to a circle, but no general definition such as that given by Casey. The earlier of the two occurs in a paper of Steiner's in the first volume of "Crelle," where "Potenz des Punktes in Bezug auf den Kreis, oder Potenz des Kreises in Bezug auf den Punkt" is defined as the difference between the square on the distance of the point from the centre, and the square on the radius, and is distinguished as the *interior* or *exterior* Power, according as the point is within or without the circle, being positive in both cases. The later improved form of the definition is to be found in Steiner's "Synthetische Geometrie," and differs from the former in replacing the interior power and exterior power by a single power, thus: *If P be the point, and A, B the points in which the circle is intersected by any line through P, then the rectangle $PA \cdot PB$ is defined as the power of the point P with reference to the circle.* This definition makes the power positive for external, negative for internal points.

Going back to Casey's definition, we find that it leaves a good deal to be desired. By inadvertence, as I suppose, the word *equation* is used for "that expression which, being equated to zero, gives the equation." Even if this correction were made, the definition would be incomplete until we had fixed the form of the equation still further. We should have to agree that in the equation $f(x, y) = 0$, the function $f(x, y)$ should be rational and integral as to x and y. And even then, for a given curve the "power" would be indeterminate to the extent of an arbitrary constant factor.

In the next two pages we find Casey interpreting his definition in two inconsistent senses, first taking $ax + by + c$ as the *power* of (x, y) as to the straight line $ax + by + c = 0$, and afterwards making a statement as to the power of a point, which is only true for the special form $x\cos\alpha + y\sin\alpha - p = 0$. Again, on p. 72 there is a statement as to the power of the point (x, y) with reference to a circle which is inconsistent with the definition to which reference is given.

Seeing that an authority like Casey has left the definition so indefinite, I feel at liberty to make suggestions as to what would be the most expedient usage of the term in question.

In the first place, a definition of Casey's type refers not simply to a line or curve, but to the particular form in which its equation is written. Would it not, therefore, be better to speak of the power of a point with respect to a curve as represented by a certain equation, or more briefly, the power of a point with reference to an equation? We could, for instance, speak of the power of (x, y) as to the equation $Ax + By + C = 0$, and the definition required would be:

The power of a point (x_1, y_1) *with respect to an equation* $f(x, y) = 0$ *is the value of* $f(x_1, y_1)$.

This might be generalised as follows: The power of $(x_1, y_1, z_1, \ldots)$ as to the equation $f(x, y, z, \ldots) = 0$ is the value of $f(x_1, y_1, z_1, \ldots)$

But, in addition to the term Power of a point as to an *equation*, we may also use the term Power of a point as to a line, curve, or locus.

The expression "Power of a point as to a locus" (where the locus might be a line, a curve, or a surface) would appropriately get a purely geometrical definition. The later form of Steiner's definition of the power of a point with reference to a circle, which is purely geometrical, will no doubt continue to hold the field.

In seeking a complete geometrical definition of *the Power of a point with respect to a straight line* we meet a further complication which does not arise in connection with the Power of a point as to the *equation*. If we define it to be the perpendicular distance, reckoned as positive when the point and the origin are on opposite sides of the line, the definition will fail in the case of a line passing through the origin.

It would be best, I think, to give the definition with reference to a *directed* straight line thus: The power of a point with reference to a directed straight line is its distance from that line to the right of a traveller walking along it in the positive direction.

To make this agree with the usual convention, we have only to add that the *direction* of the line is to be such that the origin is to the left of the line. But I think this ought to be treated as a *special* convention, and not put into the definition, for in many cases such a restricted convention is disadvantageous.

Returning now to the analytical definition of the Power of a point with reference to an *equation*, we may note the following geometrical interpretations.

Let O be the origin and P the point (x, y) and S the point in which OP cuts the line whose equation is $x/a+y/b-1=0$. The power of P with respect to the equation is equal to the ratio SP : OS.

Again, with reference to the equation $Ax+By+C=0$ the power of P is $-C \times SP : OS$.

The power of (x, y) for the equation $(x-a)^2+(y-b)^2-r^2=0$ is the rectangle PQ . PR, when PQR is any straight line passing through P and cutting in Q, R the circle whose centre is (a, b) and radius r.

The power of (x, y) for the equation $Ax^2+Ay^2+Bx+Cy+D=0$ is obviously equal to $A \times PQ . PR$.

The power of (x, y) for the equation $x^2/a^2+y^2/b^2-1=0$ is the ratio $-PQ . PR : (OD . OD')$, or $PQ . PR : OD^2$, where PQR is any secant through P, and DOD′ is a diameter parallel to it.

The power of (x, y) for the equation $y^2-4ax=0$ is the rectangle PQ . PR where PQR is a line parallel to the directrix, intersecting the parabola in Q, R; it is also equal to $4a$. PV, where PV is a line parallel to the axis of the parabola and meeting the parabola in V.

The power of (x, y) as to the general equation of the second degree,

$$u \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0,$$

is $= -\text{PQ}\,.\,\text{PR}\,.\,u_0/\text{OD}^2$; where u_0 is the power of the centre, and may be written $gx_0 + fy_0 + c$ or

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} \div \begin{vmatrix} a & h \\ h & b \end{vmatrix}.$$

The power of (x, y) as to any rational integral algebraic equation $f(x, y) = 0$ receives its interpretation through the equation

$$\frac{f(x, y)}{f(x_1, y_1)} = \frac{\text{PA}\,.\,\text{PB}\,.\,\text{PC}\ldots\ldots}{\text{P}_1\text{A}\,.\,\text{P}_1\text{B}\,.\,\text{P}_1\text{C}\ldots},$$

where A, B, C... are the points in which the line joining the points $\text{P}(x, y)$ and $\text{P}_1(x_1, y_1)$ intersects the locus of $f(x, y) = 0$.

Here (x_1, y_1) is an arbitrarily chosen point.

A similar interpretation holds in solid geometry for the power of (x, y, z) as to the equation $f(x, y, z) = 0$, if it be algebraic, rational, and integral.

The power of P, (ξ, η, ζ) as to the equation $\text{A}\xi + \text{B}\eta + \text{C}\zeta = 0$ which represents a straight line, ξ, η, ζ being any point-coordinates whose invariable relation is $\lambda\xi + \mu\eta + \nu\zeta = 1$, may be interpreted thus: $\dfrac{\text{A}\xi + \text{B}\eta + \text{C}\zeta}{\text{A}\xi_1} = \dfrac{p}{p_1}$, where p is the perpendicular from (ξ, η, ζ) on the line, and p_1 the perpendicular from $(\xi_1, 0, 0)$ one of the angular points of the fundamental triangle. Since $\lambda\xi_1 = 1$, we get the power of (ξ, η, ζ) to be $= \dfrac{\text{A}p}{\lambda p_1} = \dfrac{\text{A}ap}{2\lambda\Delta}$, where a is one side and Δ the area of the fundamental triangle.

The following application of the 'power of an equation' is given because it involves a point of interest with reference to the *sense* of the perpendicular.

It is required to write down the equations of the lines bisecting the internal angles of a triangle, the equations of the sides being given.

Let these equations be $u \equiv a\,x + by + c = 0,$

$$u' \equiv a'x + b'y + c' = 0,$$
$$u'' \equiv a''x + b''y + c'' = 0.$$

Take (x_1, y_1), (x_2, y_2), (x_3, y_3) to represent the vertices of the triangle and let $ax_1 + by_1 + c \equiv u_1$, etc.

Then we have
$$a\,x_1 + by_1 + c = u_1,$$
$$a'x_1 + b'y_1 + c' = 0,$$
$$a''x_1 + b''y_1 + c'' = 0.$$

Hence $\begin{vmatrix} a, & b, & c - u_1 \\ a', & b', & c' \\ a'', & b'', & c'' \end{vmatrix} = 0$; $\therefore \begin{vmatrix} a, & b, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{vmatrix} = u_1 \begin{vmatrix} a, & b, & 1 \\ a', & b', & 0 \\ a'', & b'', & 0 \end{vmatrix}$.

Now for points on the same side of the line $u = 0$ as the vertex P_1, u will have the same sign as u_1.

Hence $u_1(ax + by + c) \div \sqrt{u_1^2(a^2 + b^2)}$, or $u_1 u \div \sqrt{u_1^2(a^2 + b^2)}$ is the perpendicular distance of (x, y) from $u = 0$, reckoned as positive when on the same side as P_1.

Hence the equations to the in-centre of $P_1P_2P_3$ are

$$uu_1 \div \sqrt{u_1^2(a^2 + b^2)} = u'u_2' \div \sqrt{u_2'(a'^2 + b'^2)} = u''u_3'' \div \sqrt{u_3''(a''^2 + b''^2)},$$

or, explicitly in terms of the given coefficients,

$$(ax + by + c) . \begin{vmatrix} a', & b' \\ a'', & b'' \end{vmatrix} . \begin{vmatrix} a, & b, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{vmatrix} \div \sqrt{\left\{ \begin{vmatrix} a', & b' \\ a'', & b'' \end{vmatrix} . \begin{vmatrix} a, & b, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{vmatrix} . (a^2 + b^2) \right\}}$$
$$= \text{etc.} = \text{etc.}$$

Remark suggested by the above: If $ax + by + c = 0$ be the equation of a line, and we write it in the form

$$(ax_1 + by_1 + c)(ax + by + c) = 0,$$

then the power of (x, y) with respect to the latter equation will be *positive* for points on the same side of the line as the point (x_1, y_1).

Thus for $-c(ax+by+c)=0$, (x, y) has *positive* power for points on the side remote from the origin, and

$$-c(ax+by+c) \div \sqrt{(a^2+b^2)c^2} = 0 \text{ is the } \textit{standard}$$

form of the equation of the straight line.

Of course we might use here instead of the factor (ax_1+by_1+c), any power of that factor whose index is an odd integer, positive or negative, or in fact any odd function of that factor.

More general remark: To express "that value of $\pm a$ which has the same sign as b" we may use $b\sqrt{\frac{a^2}{b^2}}$, or $\frac{\sqrt{a^2b^2}}{b}$. For example, that value of ± 1 which has the same sign as b, is $b/\sqrt{b^2}$ or $\sqrt{b^2}/b$.

Note that so long as x is *real*, the symbols $\sqrt{x^2}$ and $|x|$ have the same meaning.

Solution of a Geometrical Problem.

By R. F. Muirhead.

"To draw through a given point a transversal of a given triangle so that the segments of the transversal may be in a given ratio."

Figure 25.

Analysis. ABC is a triangle and DEF a transversal and K is the point of concurrence of the four circles circumscribed about the four triangles formed by the transversal and the sides of the triangle. H is a point on the circumference of the circumcircle of ABC, such that AK and AH are equally inclined to the bisector of the angle BAC.

Thus $\qquad \angle HBC = \angle KCB = \angle KED$

and $\qquad \angle HCB = \angle KBC = \angle KFE.$

Thus the triangles HBC, KCB and KEF are mutually similar.

Again if AH meet BC in X, we have

$$\angle XHC = \angle ABC = \angle FKD.$$

Hence the figures HBCX and KEFD are similar;

hence $\qquad CX : XB = FD : DE.$

Let O be any point in EF, and OLDK a circle meeting KB in L.

Then $\qquad \angle OLK = \angle ODK = \angle FBK.$

$\therefore$ OL is parallel to FB.

Construction. Given ABC and O, and the ratio $p:q$ to which FD : DE is to be equal. Make $CX:XB = p:q$. Draw AXH meeting the circumcircle of ABC in H. Make arc BK equal and opposite to arc CH. Draw OL parallel to BA to meet KB in L, and let the circle through OLK cut BC in D and D′. Then the lines OD and OD′ give the two solutions of the problem.

Cor. If we make O go off to infinity in a given direction, the arc KDL becomes a straight line through K making $\angle$ BKD equal to the angle between BA and the given direction.

This enables us to solve the problem : *To draw a transversal of a triangle in a given direction so that the segments of the transversal may be in a given ratio.*

Remark. We may note that K is the focus of the parabola which touches the three sides of ABC and the transversal, and that all other transversals (not passing through O) which have their segments in the same ratio, will touch the same parabola. Hence we see that if three fixed tangents be drawn to a parabola, the ratio of the segments they intercept on any variable tangent is a fixed one. This is a known property of tangents to a parabola. (See Professor Gibson's paper in Vol. IX. of our *Proceedings.*)

From this it follows that we could reduce our problem to that of drawing through a given point a fifth tangent to a parabola when four are given. For, ABC being given, and the ratio $p:q$, we can by a very easy construction draw a transversal (say through an arbitrary point in AB) having its segments in the ratio $p:q$. We have then four given tangents.

On the other hand our construction affords a new solution of the problem : to draw through a given point a tangent to the parabola which touches four given lines.

The only previously published solution I have met with of the problem of this paper is that given in Thomas Simpson's *Elements of Geometry* (Problem XXXVII. of the section on the Construction of Geometrical Problems). The solution there given is not at all direct or elegant : a footnote gives a reference to David Gregory's *Astronomy*, B.V. Prob. 8.

I ought to add that the idea of the solution here given was suggested to me by the solution of the special case of the problem when $p:q=1$ which was communicated to me by a friend.

Minimum Deviation through a Prism, etc.

By R. F. Muirhead.

Let $\mu \equiv$ refractive index of the prism.

Let α, α', β', β be the successive angles of incidence and refraction of a ray, in a plane perpendicular to the edge of the prism.

$$\text{Then} \quad \left.\begin{array}{l} \sin\alpha = \mu\sin\alpha' \\ \sin\beta = \mu\sin\beta' \end{array}\right\} \qquad (1)$$

Consider the case when $\mu > 1$, so that $\alpha > \alpha'$ and $\beta > \beta'$. $\qquad$ (2)

First Method. (Fig. 26.)

Let O be the centre of a circle of unit radius and let $\angle$ XOA' and $\angle$ XOB' on opposite sides of OX be the angles α' and β' respectively, so that $\angle$ A'OB' $= i$, the angle of the prism.

Take OA' = OB' $= \mu$.

Draw A'A, B'B parallel to XO to meet the circle in A, B.

Join OA, OB.

We have $\quad \sin OAA'/\sin AA'O = OA'/AO = \mu$;

$\therefore \angle AOX = \alpha$. $\quad$ Similarly $\angle BOX = \beta$.

Suppose now $\alpha' \neq \beta'$, say $\alpha' > \beta'$, then $\angle AA'O > \angle BB'O$.

Hence, since A' and B' are equally distant from O, we have, by a slight extension of *Euc.* III. 8, A'A > B'B.

From AA' cut off AK = BB' and join AB, B'K, so that ABB'K is a parallelogram.

Now $\quad \angle AA'O > \angle BB'O$ and $\angle OA'B' = \angle OB'A'$;

$$\begin{aligned} \therefore\ \angle AA'B' &> \angle BB'A' > \angle BB'K \\ &> \angle B'KA'; \end{aligned}$$

$$\therefore\ B'K > B'A';$$

$$\therefore\ BA > B'A'.$$

On the other hand, if $\alpha' = \beta'$, it is obvious that $AB = A'B'$.

Now, for a given prism, $A'B'$ is a fixed length, since $OA' = OB' = \mu$, and $\angle A'OB' = i$.

Hence when $\alpha' = \beta'$, AB is a minimum ; $\therefore \angle AOB$ (or $\alpha + \beta$) is a minimum, $\therefore\ \alpha + \beta - i$ is a minimum.

Thus the deviation is least when $\alpha' = \beta'$.

Second Method. (Fig. 27.)

Let ADB be a circle of unit radius and centre O.

Let $\quad CO = \mu, \quad \angle OCB = \beta'$ and $\angle OCA = \alpha'$;

$\therefore \quad \angle AOC = \alpha - \alpha'$ and $\angle BOC = \beta - \beta'$.

Let Q be the centre of the circle circumscribing ABC.

Let OQ meet this circle in E and the other in D.

Join CD, CE.

Thus $\angle OCE = \dfrac{\alpha' + \beta'}{2} = \dfrac{i}{2}$ and $\angle COD = \dfrac{\alpha + \beta - \alpha' - \beta'}{2} = \dfrac{\delta}{2}$,

where $\delta \equiv$ deviation of ray.

Let CE meet the arc ADB in F.

As α' approaches equality with β', A, B and E tend towards coincidence at F. Hence, since F, C, O are fixed points, the angle COE or $\delta/2$ is least in the limiting case when E coincides with F, *i.e.*, when $\alpha' = \beta'$.

Third Method. (Fig. 28.)

This is a modification of the second.

Let O, C, A, B, F be the same as before, so that CF bisects $\angle BCA$.

Join AF, and produce BF to meet CA in G.

Now $\angle CBF$ is obtuse, $\therefore\ CB < CF < CG$.

But $BF : FG = BC : CG$,

$\therefore\ BF < FG$

$\qquad < FA$, since $\angle FGA$ is obtuse.

Hence $\angle BOF < \angle FOA$; $\therefore \angle COB + \angle COA >$ twice $\angle COF$.

But twice $\angle COF$ is the value of δ when $\alpha' = \beta'$.

Hence in all other cases δ is greater.

Corollary. We also see from Fig. 27 or Fig. 28 that $\alpha - \alpha'$, the deviation due to a *single* refraction increases as α' increases uniformly, and at an increasing rate. For if we take OCB, OCF and OCA as three successive values of α' increasing by equal increments BCF, FCA, the corresponding deviations increase by the amounts BOF, FOA, of which the latter is the greater.

It may be noted that the Second and Third methods could be modified by taking $OC = 1$ and the radius $OA = \mu$ and interchanging α, α' and β, β'. And from the modified figure we could deduce, as in the previous corollary, that as α increases uniformly, the deviation $\alpha - \alpha'$ increases at an increasing rate.

In Heath's *Treatise* there are proofs of these results by the use of infinitesimals, ascribed to the late Professor Tait.

The following propositions in Geometry, amongst others that could be stated, are corollaries to what precedes :—

I. If from two fixed points without a fixed circle, and equi-distant from its centre, two parallel lines be drawn cutting the circle, the equal arcs intercepted by them on the circumference are *least* when the parallel lines are equi-distant from the centre.

II. If from a fixed point without a fixed circle, a pair of lines including an angle of fixed size are drawn to cut the circle, the two arcs intercepted between them on the circumference are both *least* when the lines are equi-distant from the centre.

III. If from the centre of a fixed circle two radii are drawn, including an angle fixed in size, then the angle subtended at a fixed external point by the arc between the extremities of the radii is *greatest* when the radii are equally distant from the fixed point.

In seeking for a concise *trigonometrical* proof of the minimum deviation theorem for the prism, I arrived at a formula, which I found to be none other than that given in Parkinson's Optics. It is one that seems to leave nothing to be desired in the way of conciseness and neatness, but its popularity has perhaps suffered from the rather unsymmetrical and difficult way in which Parkinson deduces it.

I recall it here partly in order to indicate a simpler proof of the formula, and partly to associate with it a companion formula which enables us to deduce the theorem as to a single refraction given as a corollary above.

If α, α', β', β have the same significations as before, and we wish to deduce results connecting $i = \alpha' + \beta'$ and $\delta = \alpha + \beta - \alpha' - \beta'$ from the law of refraction, which gives $\sin\alpha = \mu\sin\alpha'$, $\sin\beta = \mu\sin\beta'$; let us consider the formulae

$$\begin{aligned} \cos(\alpha-\beta) - \cos(\alpha+\beta) &= 2\sin\alpha\sin\beta \\ &= 2\mu^2\sin\alpha'\sin\beta' \\ &= \mu^2\{\cos(\alpha'-\beta') - \cos(\alpha'+\beta')\}; \\ \cos(\alpha-\beta)\cos(\alpha+\beta) - 1 &= -\sin^2\alpha - \sin^2\beta \\ &= -\mu^2(\sin^2\alpha' + \sin^2\beta') \\ &= \mu^2\{\cos(\alpha'-\beta')\cos(\alpha'+\beta') - 1\}. \end{aligned}$$

Putting

$$x \equiv \cos(\alpha-\beta),\ y \equiv \cos(\alpha+\beta),\ z \equiv \cos(\alpha'-\beta'),\ c \equiv \cos(\alpha'+\beta') = \cos i,$$

we may write the above results thus :—

$$x - y = \mu^2(z - c),$$

$$xy - 1 = \mu^2(zc - 1).$$

Eliminating z, we get $(c+x)(c-y) = (\mu^2-1)(1-c^2) = (\mu^2-1)\sin^2 i$, which is Parkinson's formula.

Eliminating c, we get $(z-x)(z+y) = (\mu^2-1)(1-z^2)$.

From the latter formula, if we suppose z constant, we find that x decreases with y, $\therefore$ $\alpha - \beta$ increases with $\alpha + \beta$. Thus, for a given value of $\alpha' - \beta'$, if β increases so does β'; $\therefore$ α' increases; $\therefore$ α increases; $\therefore$ $\alpha + \beta$ increases; $\therefore$ $\alpha - \beta$ increases.

Note on a Theorem of Lommel.

By F. H. Jackson, M.A.

1.

Among the many formulae which show special relations existing between the circular functions and the Bessel-Function $J_n(x)$, when n is half an odd integer, there is one due to Lommel

$$\frac{\sin 2x}{\pi} = \{J_{\frac{1}{2}}(x)\}^2 - 3\{J_{\frac{3}{2}}(x)\}^2 + 5\{J_{\frac{5}{2}}(x)\}^2 - \ldots\ldots$$

In connection with the paper on Basic sines and cosines in this volume of the *Proceedings*, it may be interesting to consider briefly an analogue of Lommel's theorem, which we write

$$\sum_{r=0}^{r=\infty}(-1)^r\frac{[4r+2]}{[2]}p^{r(r-1)}J_{\left[\frac{2r+1}{2}\right]}(x)\,\mathfrak{J}_{\left[\frac{2r+1}{2}\right]}(x)$$

$$=\frac{1}{[2]^2\{\Gamma_{p^2}[\frac{3}{2}]\}^2}\left[(1+p)x-\frac{2(1+p)(1+p^3)x^3}{[3]!}+\frac{2(1+p^2)(1+p)(1+p^3)(1+p^5)x^5}{[5]!}-\ldots\right], \quad (2)$$

the general term of the series within the large brackets being

$$(-1)^r\frac{2(1+p^2)(1+p^4)\ldots(1+p^{2r-2})\,.\,(1+p^1)(1+p^3)\ldots(1+p^{2r+1})}{[2r+1]!}x^{2r+1}.$$

When the base p equals 1, this series reduces to $\dfrac{\sin 2x}{\pi}$.

2.

Defining $J_{[n]}(x)$ as $\displaystyle\sum_{r=0}^{r=\infty}(-1)^r\frac{x^{n+2r}}{\{2n+2r\}!\,\{2r\}!}$,

$\{2m\}!$ is in general $[2]^m\,\Gamma_{p^2}([m+1])$.

This reduces, when m is a positive integer, to $[2][4][6]\ldots[2m]$.

We take $\quad \displaystyle\mathfrak{J}_{[n]}(x) = \sum_{r=0}^{r=\infty}(-1)^r\frac{x^{n+2r}}{\{2n+2r\}!\,\{2r\}!}p^{2r(n+r)}.$

This function is connected with $J_{[n]}$ by an inversion of the base p.

In a paper shortly to be printed (*Proc. R. S.*) it is shown that

$$J_{[m]}(x)\,\mathfrak{J}_{[n]}(x) = J_{[n]}(x)\,\mathfrak{J}_{[m]}(x) = \sum_{r=0}^{r=\infty}(-1)^r \frac{\{2m+2n+4r\}_r}{\{2m+2r\}\,!\,\{2n+2r\}\,!\,\{2r\}\,!}\,x^{m+n+2r}, \quad (3)$$

where $\{2m+2n+4r\}_r = [2m+2n+4r][2m+2n+4r-2]\ldots[2m+2n+2r+2]$.

3.

Consider now the series

$$[2]J_{[\frac{1}{2}]}\mathfrak{J}_{[\frac{1}{2}]} - [6]J_{[\frac{3}{2}]}\mathfrak{J}_{[\frac{3}{2}]} + \ldots + (-1)^s p^{s(s-1)} J_{[\frac{2s+1}{2}]}\mathfrak{J}_{[\frac{2s+1}{2}]} - \ldots\ldots \quad (4)$$

This series may be written by means of (3) in the form

$$[2]\Sigma(-1)^r \frac{\{4r+2\}_r}{\{2r+1\}\,!\,\{2r+1\}\,!\,\{2r\}\,!}\,x^{2r+1}$$

$$-[6]\Sigma(-1)^r \frac{\{4r+6\}_r}{\{2r+3\}\,!\,\{2r+3\}\,!\,\{2r\}\,!}\,x^{2r+3}$$

$$\ldots$$

$$\ldots$$

$$(-1)^{r+s}[4s+2]\,\Sigma\,\frac{\{4r+4s+2\}_r}{\{2r+2s+1\}\,!\,\{2r+2s+1\}\,!\,\{2r\}\,!}\,x^{2r+2s+1}.$$

$$\ldots$$

$$\ldots$$

Collecting the terms in a series of ascending powers of x, the coefficient of x arises only from the first of these series, and is

$$[2]\frac{\{2\}_0}{\{1\}\,!\,\{1\}\,!\,\{0\}\,!},$$

which reduces to

$$\frac{[2]}{[2]^{\frac{1}{2}}[2]^{\frac{1}{2}}\{\Gamma_{p^2}(\frac{3}{2})\}^2}$$

$$= \frac{1}{\{\Gamma_{p^2}([\frac{3}{2}])\}^2},$$

as is seen from the definition of the function $\{2n\}\,!$.

The coefficient of x^3 is

$$-[2]\frac{\{6\}_1}{\{3\}\,!\,\{3\}\,!\,\{2\}\,!} - [6]\frac{\{6\}_0}{\{3\}\,!\,\{3\}\,!\,\{0\}\,!}$$

and this reduces to

$$\frac{2(1+p^2)}{[3]\,!}\,\frac{1}{\{\Gamma_{p^2}[\frac{3}{2}]\}^2}.$$

4.

The reduction, however, of the series which forms the coefficient of x^{2r+1}, offers some difficulty and is effected by a theorem due to Heine (*Kugelfunctionen*, Appendix to Chap. II., Vol. I.).

The coefficient of x^{2r+1} is, after some obvious reductions,

$$(-1)^r \frac{\{4r+2\}_r}{\{2r+1\}!\{2r+1\}!\{2r\}!}\left[[2]+[6]\frac{[2r]}{[2r+4]}+p^2[10]\frac{[2r][2r-2]}{[2r+4][2r+6]}+\ldots\right.$$

$$\left.+p^{s(s-1)}[4s+2]\frac{[2r][2r-2]..[2r-2s+2]}{[2r+4]\ldots\ldots[2r+2s+2]}+\ldots\right]. \qquad - \qquad - \qquad (5)$$

Now $\quad [2]=\dfrac{p^2-1}{p-1}=\dfrac{p^2}{p-1}-\dfrac{1}{p-1},$

$[6]=\dfrac{p^6}{p-1}-\dfrac{1}{p-1},$

......

......

$[4s+2]=\dfrac{p^{4s+2}}{p-1}-\dfrac{1}{p-1},$

......

......

therefore we write the series which is within the large brackets of expression (5) as the difference of two series, viz.,

$$\frac{p^2}{p-1}S_1-\frac{1}{p-1}S_2=\frac{p_2}{p-1}\left\{1+p^4\frac{[2r]}{[2r+4]}+..+p^{s(s+3)}\frac{[2r].....[2r-2s+2]}{[2r+4]..[2r+2s+2]}+..\right\}$$

$$-\frac{1}{p-1}\left\{1+\frac{[2r]}{[2r+4]}+..+p^{s(s-1)}\frac{[2r].....[2r-2s+2]}{[2r+4]..[2r+2s+2]}+..\right\}. \quad (6)$$

5.

Heine has shown that if

$$1+\frac{(1-a)(1-b)}{(1-q)(1-c)}x+\frac{(1-a)(1-aq)(1-b)(1-bq)}{(1-q)(1-q^2)(1-c)(1-cq)}x^2+\ldots=\phi[a,\,b,\,c,\,q,\,x],$$

then $\phi[a,\,b,\,c,\,q,\,x]=\displaystyle\prod_{n=0}^{n=\infty}\frac{(1-bxq^n)\left(1-\frac{c}{b}q^n\right)}{(1-xq^n)(1-cq^n)}\cdot\phi\left[b,\,\frac{abx}{c},\,bx,\,q,\,\frac{c}{b}\right].$

If in this transformation we put
$$a = p^2,\quad q = p^2,\quad b = p^{-2r},\quad c = p^{2r+4},\quad x = -p^{2r+4},$$
$\phi[a, b, c, q, x]$ becomes, after simple and obvious reductions, identical with S_1.

The infinite product on the right side of Heine's transformation, reduces (r integral) to the finite product
$$\frac{(1+p^4)(1+p^6)\ldots\ldots(1+p^{2r+2})}{(1-p^{2r+4})(1-p^{2r+6})\ldots(1-p^{4r+2})}$$
which we will for convenience denote by P_1.

$$\phi\left[b, \frac{abx}{c}, bx, q, \frac{c}{b}\right] \text{ becomes}$$

$$\left\{1 - \frac{(p^{2r}-1)(p^{2r-2}+1)}{(p^2-1)(p^4+1)}p^6 + \frac{(p^{2r}-1)(p^{2r-2}-1)(p^{2r-2}+1)(p^{2r-4}+1)}{(p^2-1)(p^4-1)(p^4+1)(p^6+1)}p^{16} - \ldots\right\}$$
$$= 1 - a_1 + a_2 - \ldots,$$
so that
$$S_1 = P_1\{1 - a_1 + a_2 - a_3 + \ldots\}.$$

Similarily, if we put
$$a = p^2,\quad q = p^2,\quad b = p^{-2r},\quad c = p^{2r+4},\quad x = -p^{2r},$$
in Heine's transformation, we obtain
$$\phi[a, b, c, q, x] = S_2,$$
$$\phi\left[b, \frac{abx}{c}, bx, q, \frac{c}{b}\right]$$
$$= \left\{1 - \frac{(p^{2r}-1)(p^{2r+2}+1)}{(p^2-1)(1+1)}p^2 + \frac{(p^{2r}-1)(p^{2r-2}-1)(p^{2r+2}+1)(p^{2r}+1)}{(p^2-1)(p^4-1)(1+1)(1+p^2)}p^8 - \ldots\right\}$$
$$= 1 - b_1 + b_2 - \ldots.$$

The infinite product becomes
$$\frac{2(1+p^2)(1+p^4)\ldots(1+p^{2r-2})}{(1-p^{2r+4})(1-p^{2r+6})\ldots(1-p^{4r+2})} = P_2.$$

Finally we have

$$\frac{p^2}{p-1}S_1-\frac{1}{p-1}S_2=\frac{p^2}{p-1}P_1\{1-a_1+a_2-\ldots\}-\frac{1}{p-1}P_2\{1-b_1+b_2-\ldots\}. \tag{7}$$

P_1 and P_2 have most of their factors in common, so taking out the common part we may write (7)

$$\frac{(1+p^4)(1+p^6)\ldots(1+p^{2r-2})}{(1-p^{2r+4})(1-p^{2r+6})\ldots(1-p^{4r+2})}\cdot\frac{1}{(p-1)}$$

$$\left\{p^2(1+p^{2r})(1+p^{2r+2})\{1-a_1+a_2-\ldots\}-2(1+p^2)\{1-b_1+b_2-\ldots\}\right\}. \tag{8}$$

We sum the series within the large brackets as follows:

$$\begin{aligned}&-2(1+p^2)\\ &+2(1+p^2)b_1+p^2(1+p^{2r})(1+p^{2r+2})\\ &-2(1+p^2)b_2-p^2(1+p^{2r})(1+p^{2r+2})a_1\\ &\quad\ldots\qquad\qquad\ldots\\ &\quad\ldots\qquad\qquad\ldots\end{aligned}$$

in general taking the term involving b_n with the term involving b_{n-1}.

Without difficulty, even in the general term, we can reduce this series to

$$-2(1+p^2)+2p^2\frac{[4r+4]}{[2]}-2p^8\frac{[4r+4][4r]}{[2][8]}+2p^{18}\frac{[4r+4][4r][4r-4]}{[2][8][12]}-\ldots$$

$$=-2(1+p^2)\left[1-p^2\frac{[4r+4]}{[4]}+p^8\frac{[4r+4][4r]}{[4][8]}-\ldots\right].$$

The product expression for the series within the large brackets is

$$(1-p^2)(1-p^6)(1-p^{10})\ldots(1-p^{4r+2}).$$

Expression (8) is thus reduced to

$$-\frac{(1+p^4)\ldots(1+p^{2r-2})}{(1-p^{2r+4})\ldots(1-p^{4r+2})}\cdot\frac{(p+1)}{(p^2-1)}2(1+p^2)\,.\,(1-p^2)(1-p^6)\ldots(1-p^{4r+2}).$$

The coefficient of x^{2r+1} is then by (5) and (9), after cancelling common factors,

$$(-1)^r\frac{2(1+p^2)(1+p^4)\ldots(1+p^{2r-2})\,.\,(1+p^3)\ldots(1+p^{2r+1})}{[2r+1]!}\frac{1}{\{\Gamma_{p^2}([\tfrac{3}{2}])\}^2},$$

which establishes

$$[2]J_{[\frac{1}{2}]}\mathfrak{J}_{[\frac{1}{2}]} - [6]J_{[\frac{3}{2}]}\mathfrak{J}_{[\frac{3}{2}]} + \dots = \frac{1}{\{\Gamma_{p^2}(\frac{3}{2})\}^2}\left[x - \frac{2(1+p^3)}{[3]}x^3 + \dots\right].$$

Dividing throughout by [2] we have the theorem (2) as stated.

We may show also that

$$J_{[\frac{1}{2}]}\mathfrak{J}_{[\frac{1}{2}]} + J_{[-\frac{1}{2}]}\mathfrak{J}_{[-\frac{1}{2}]} = \frac{[2]}{x\{\Gamma_{p^2}(\frac{1}{2})\}^2},$$

which reduces when $p = 1$ to

$$J_{\frac{1}{2}}^2 + J_{-\frac{1}{2}}^2 = \frac{2}{\pi x}, \quad \text{(Lommel)}$$

and, in general, if $n = \frac{1}{2}(2\kappa + 1)$,

$$J_{[n]}\mathfrak{J}_{[n]} + J_{[-n]}\mathfrak{J}_{[-n]} = \frac{1}{\{\Gamma_{p^2}(\frac{1}{2})\}^2}[a_1x^{-1} + a_2x^{-2} + \dots + a_{\kappa-1}x^{-2\kappa+1}],$$

where a_1, $a_2, \dots$ are simple expressions of factors of the type $[m]$.

Seventh Meeting, 10th June 1904.

Mr CHARLES TWEEDIE, President, in the Chair.

The Turning-Values of a Cubic Function and the nature of the roots of a Cubic Equation.

By P. PINKERTON, M.A.

The first part of this paper depends on the theorem that if a, b, c are three positive quantities such that

$$a + b + c = \text{a constant},$$

then abc is a maximum when $a = b = c$; with the corollary that if a, b, c are three negative quantities such that

$$a + b + c = \text{a constant},$$

then abc is a minimum when $a = b = c$.

1. Consider the graph of

$$y = x(x - a)^2,$$

where a is positive.

The graph meets OX at the points (0, 0) (a, 0). The graph has a minimum point at (a, 0), for on shifting the origin to (a, 0) the equation becomes $$y = (\xi + a)\xi^2,$$
and a first approximation at the new origin is

$$y = a\xi^2,$$

so that the graph close to that point is of the form of a festoon. There is also a maximum point in the interval between $x = 0$ and $x = a$. To determine the point we observe that $x(x - a)^2$ is a maximum when $2x(a - x)(a - x)$ is a maximum. Now each of these factors is positive in the interval considered and their sum is constant ($= 2a$) ; $\therefore$ a maximum value occurs when

$$2x = a - x \quad i.e., \text{ when } x = \frac{a}{3}.$$

The maximum value is therefore $\dfrac{a}{3}\left(\dfrac{a}{3} - a\right)^2 = \dfrac{4a^3}{27}$.

2. Again considering the graph of

$$y = x(x + a)^2,$$

where a is positive, we observe that if the origin is shifted to the point $(-a, 0)$ a first approximation at the new origin is

$$y = -a\xi^2$$

which represents an inverted festoon. There is therefore a maximum point at $(-a, 0)$ and a minimum point in the interval $x = 0$ to $x = -a$. To find this point we observe that $x(x+a)^2$ has its minimum value when $2x(-a-x)(-a-x)$ is a minimum. Each of these factors is negative in the interval considered and their sum is constant $(= -2a)$;

$$\therefore \quad 2x = -a - x \text{ for the minimum point;}$$

$$\therefore \quad x = -\frac{a}{3};$$

and the minimum value is $-\dfrac{4a^3}{27}$.

3. In general let

$$y = x^3 + px^2 + qx + r = (x+a)(x+b)^2 + c.$$

Equating coefficients, we have

$$p = a + 2b, \qquad (1)$$

$$q = 2ab + b^2, \qquad (2)$$

$$r = ab^2 + c. \qquad (3)$$

Eliminating a from (1) and (2),

$$3b^2 - 2bp + q = 0;$$

$\therefore$ real values of a, b, c can be found if

$$p^2 \geqq 3q.$$

If $p^2 = 3q$, it is clear that we can write

$$y = x^3 + px^2 + qx + r = \left(x + \frac{p}{3}\right)^3 + \left(r - \frac{p^3}{27}\right),$$

and by changing the origin to $\left(-\dfrac{p}{3},\ r - \dfrac{p^3}{27}\right)$ the equation takes the form $\eta = \xi^3$ which has a point of inflexion at the new origin and no turning-points.

If $p^2 > 3q$, $y=(x+a)(x+b)^2+c$ where a, b, c are real.
Shift the origin to $(-a, c)$ and the equation becomes

$$\eta=\xi(\xi-\overline{a-b})^2.$$

Hence, if $a-b$ is positive, there is a minimum turning-point at

$$\xi=a-b,\ \eta=0\ \textit{i.e.}, \text{ at } x=-b,\ y=c.$$

Also there is a maximum turning-point at

$$\xi=\tfrac{1}{3}(a-b),\ \eta=\tfrac{4}{27}(a-b)^3$$

$$\textit{i.e.}, \text{ at } x=-a+\tfrac{1}{3}(a-b),\ y=c+\tfrac{4}{27}(a-b)^3.$$

If $(a-b)$ is negative, there is a maximum turning-point at

$$\xi=a-b,\ \eta=0\ \textit{i.e.}, \text{ at } x=-b,\ y=c$$

and a minimum turning-point at

$$\xi=\tfrac{1}{3}(a-b),\ \eta=\tfrac{4}{27}(a-b)^3$$

$$\textit{i.e.}, \text{ at } x=-a+\tfrac{1}{3}(a-b),\ y=c+\tfrac{4}{27}(a-b)^3.$$

The graph whose equation is

$$y=\mathrm{A}x^3+px^2+qx+r$$

can clearly be reduced to the above case.

4. The nature of the roots of a cubic equation can be deduced from our knowledge of the above graphs.

Suppose the equation first brought to the form

$$x^3+qx+r=0.$$

Let $\qquad y=x^3+qx+r=(x+a)(x+b)^2+c.$

Here
$$\begin{aligned} a+2b&=0, && (1') \\ 2ab+b^2&=q, && (2') \\ ab^2+c&=r. && (3') \end{aligned}$$

If two roots are equal then clearly $c=0$;

$\therefore$ by (1′) and (2′) $\qquad -3b^2=q,$

and by (1′) and (3′) $\qquad -2b^3=r$;

$$\therefore\quad 4q^3+27r^2=0.$$

If the three roots are unequal $c \neq 0$.

In this case it is clear that if a_1, b_1, c_1 and a_2, b_2, c_2 are the two sets of solutions of (1′), (2′), (3′), then

$$a_1+a_2=0,\ b_1+b_2=0;$$

$$\therefore\quad (a_1-b_1)=-(a_2-b_2).$$

Suppose $a_1 - b_1$ to be positive and write

$$y = (x + a_1)(x + b_1)^2 + c_1.$$

It follows from the above that the minimum turning value is given by $y = c_1$.

Next, writing $\quad y = (x + a_2)(x + b_2)^2 + c_2,$

we observe that the maximum turning value is given by

$$y = c_2;$$

$\therefore$ OX will cut the graph of

$$y = x^3 + qx + r$$

in three different real points if $\dfrac{c_1}{c_2}$ is negative,

and in one real point and two imaginary points if $\dfrac{c_1}{c_2}$ is positive.

There are therefore three different real solutions of the equation, or one real and two imaginary,

	according as	$\dfrac{c_1}{c_2}$	is negative or positive,	
$\therefore$	,, ,,	$\dfrac{c_1^2}{c_1c_2}$	,, ,, ,, ,,	
$\therefore$	,, ,,	c_1c_2	,, ,, ,, ,,	
$\therefore$	,, ,,	$(r - a_1b^2)(r - a_2b^2)$	,, ,, ,, ,,	
$\therefore$	,, ,,	$r^2 + a_1a_2b^4$	,, ,, ,, ,,	$(\because a_1 + a_2 = 0)$
$\therefore$	,, ,,	$r^2 - a^2b^4$	,, ,, ,, ,,	
$\therefore$	,, ,,	$r^2 - 4b^6$	,, ,, ,, ,,	by (1′)
$\therefore$	,, ,,	$r^2 + \frac{4}{27}q^3$	,, ,, ,, ,,	by (1′) and (2′)

$\therefore$ according as $4q^3 + 27r^2$ is negative or positive.

Note on the Problem: To draw through a given point a transversal to (*a*) a given triangle (*b*) a given quadrilateral so that the intercepted segments may have (*a*) a given ratio (*b*) a given cross ratio.

By P. Pinkerton, M.A.

Note on Newton's Theorem of Symmetric Functions.

By Charles Tweedie.

§ 1. The proofs usually given that $S_n = a^n + \beta^n + \ldots$ can be expressed in terms of the elementary symmetric functions Σa, $\Sigma a\beta$, etc., though simple, are not in general elementary. The following demonstration will, I think, be found to combine these two qualities.

§ 2. If we consider any integral, homogeneous, symmetric function of degree n, its expression can be effected by the use of Typical Terms, or Types, and the Σ notation. Since these types can not contain more than n letters, it follows that the number of types can not be increased by taking more letters than n. Thus the general homogeneous, symmetric function of the third degree is

$$A\Sigma a^3 + B\Sigma a^2\beta + C\Sigma a\beta\gamma.$$

The correct expression for a smaller number of letters may be found by equating a convenient number of the original letters to zero.

It may also be remarked that any identity established among functions of degree n for a number of letters equal to n, will also hold for a greater number of letters. For we can not increase the number of types and those which do occur can not have their coefficients altered. (Some of the coefficients may, however, be zero.)

e.g., $(\Sigma ab)^2 = \Sigma a^2b^2 + 2\Sigma a^2bc + 6\Sigma abcd$ for four or more letters.

§ 3. The ordinary equations for S_1, S_2, etc., may now be readily established.

Let $\Sigma a = -p_1$; $\Sigma a\beta = p_2$; etc., as for the n letters a, β, γ, etc.

Then the equation

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots\ldots + p_n = 0 \qquad \text{(A)}$$

has for roots a, β, γ, etc.

Hence $$a^n + p_1 a^{n-1} + \ldots\ldots + p_n = 0, \qquad (1)$$
$$\beta^n + p_1 \beta^{n-1} + \ldots\ldots + p_n = 0, \qquad (2)$$
etc. etc.

Add these n identities and there results
$$S_n + p_1 S_{n-1} + \ldots\ldots + np_n = 0. \qquad \text{(B)}$$

The *same* identity is true for any number of letters greater than n. For $n - r$ letters put $p_{n-r+1} = 0, \ldots\ldots p_n = 0$.

We thus have the following identities as particular cases, true for n or more letters:
$$S_1 + p_1 \quad = 0,$$
$$S_2 + p_1 S_1 + 2p_2 = 0,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$S_n + p_1 S_{n-1} + \ldots + np_n = 0.$$

For three letters put $p_4 = 0, \; p_5 = 0, \; \ldots \; p_n = 0$, etc.

Edinburgh Mathematical Society.

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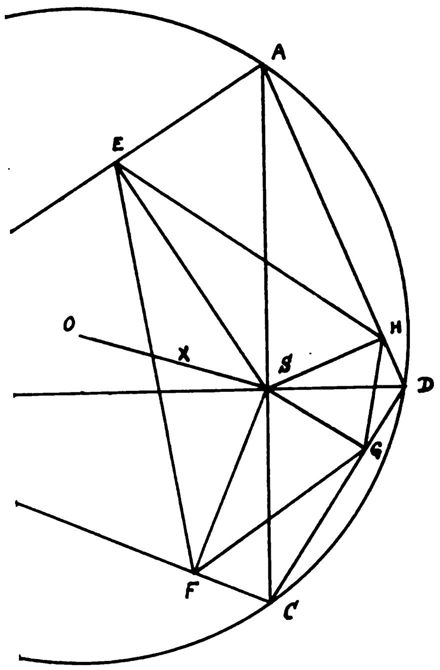

Fig. 1.

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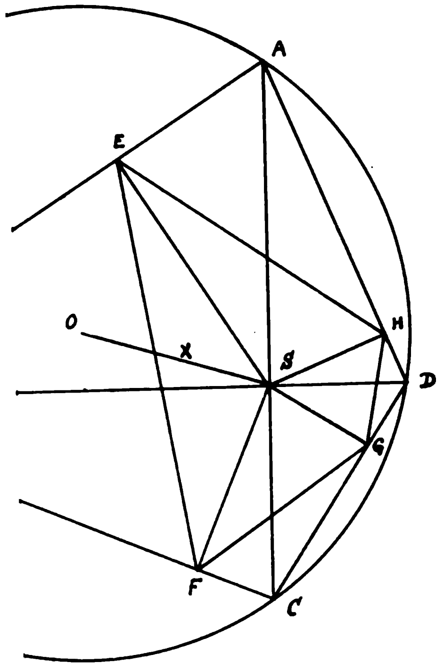

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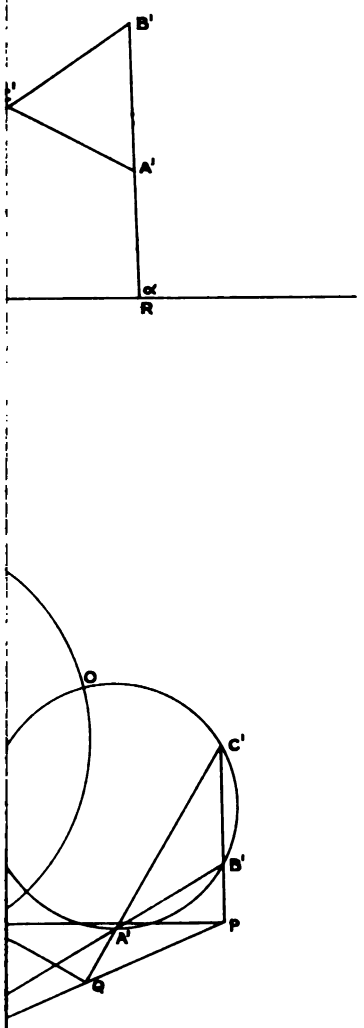

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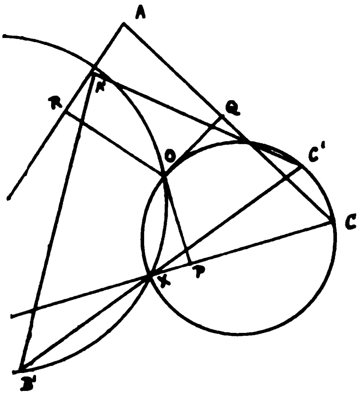

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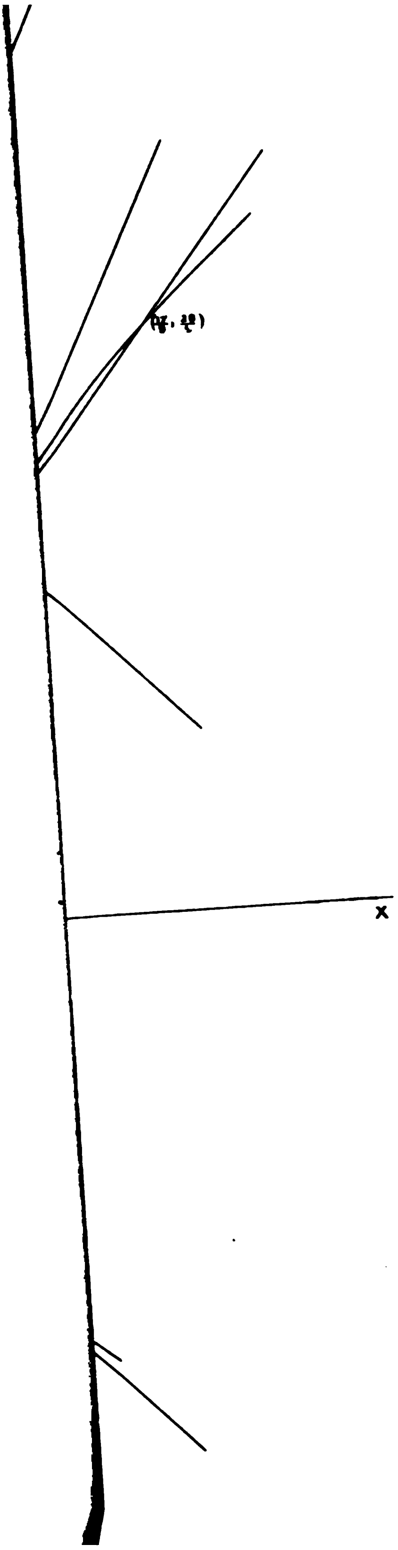
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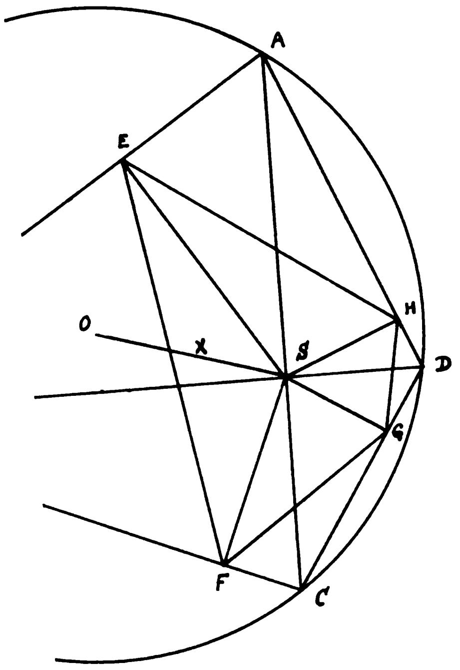

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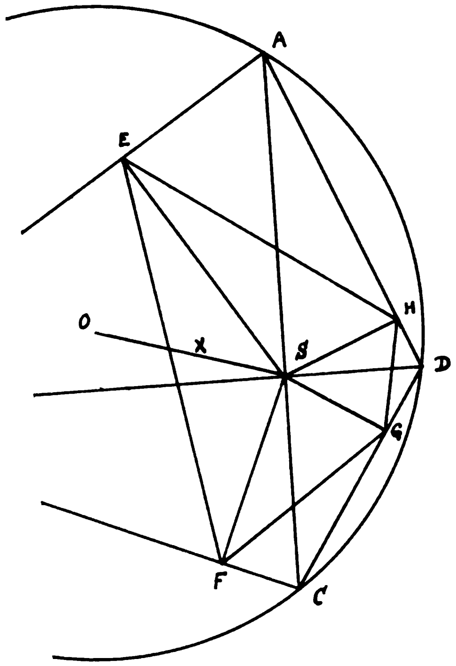

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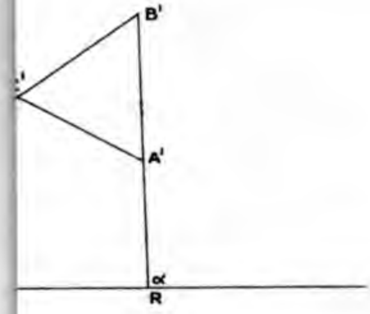

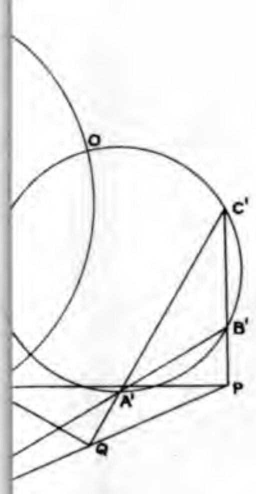

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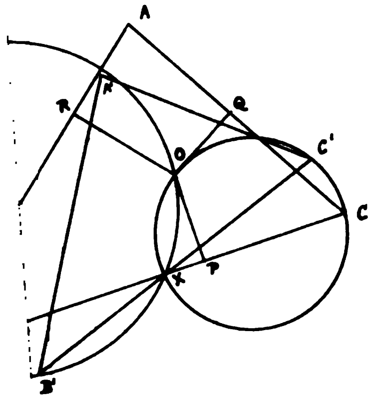

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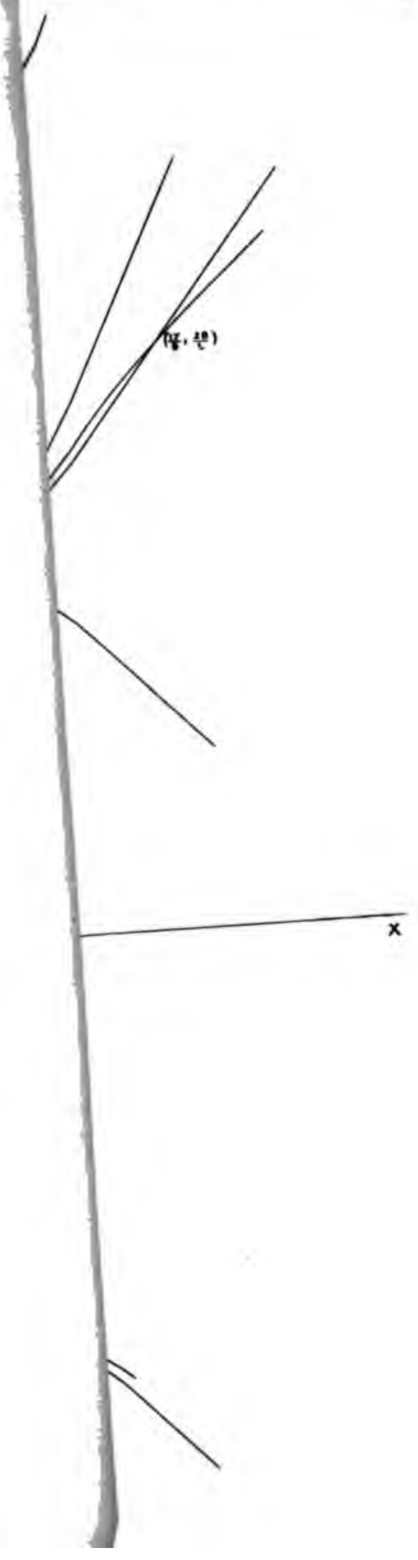

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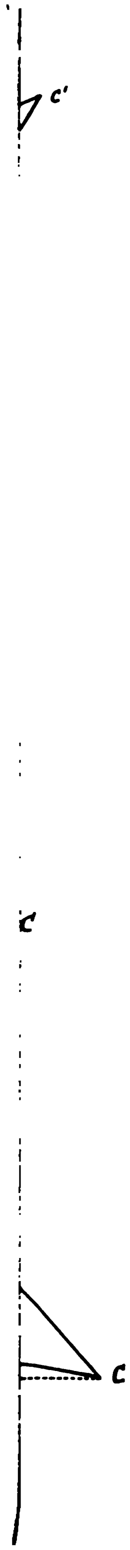

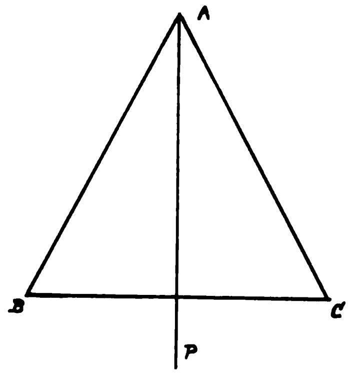

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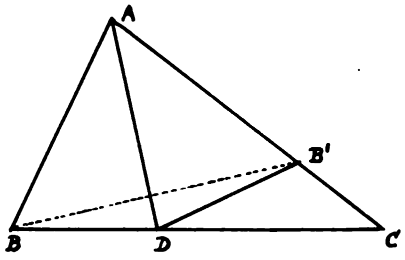

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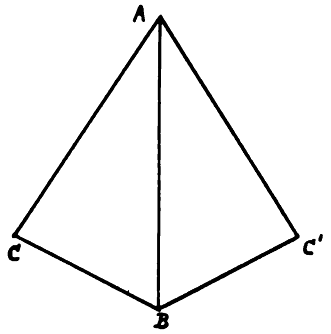

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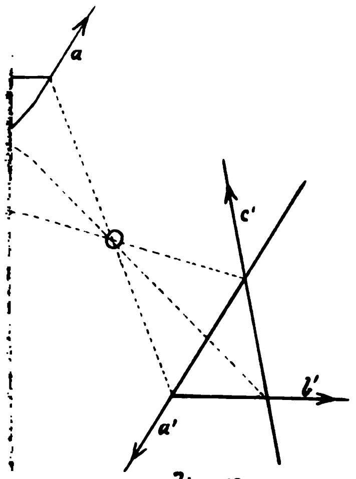

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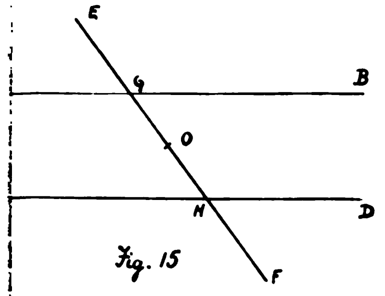

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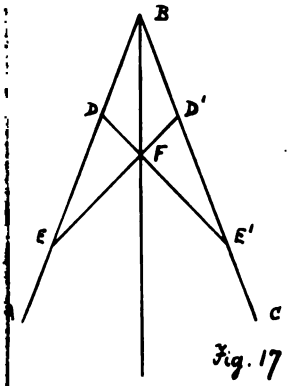

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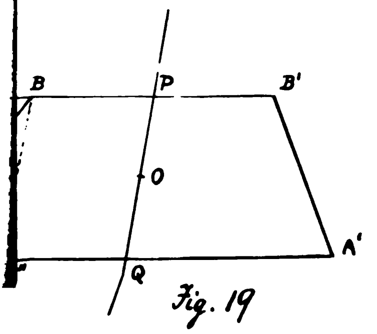

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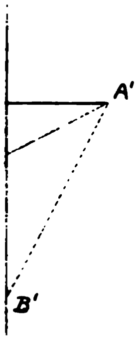

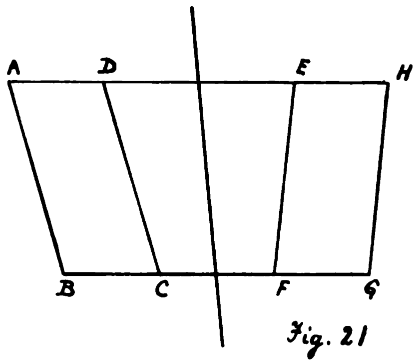

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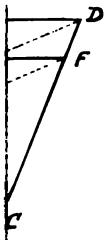

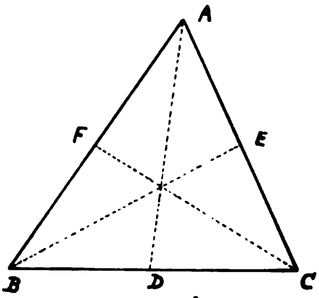

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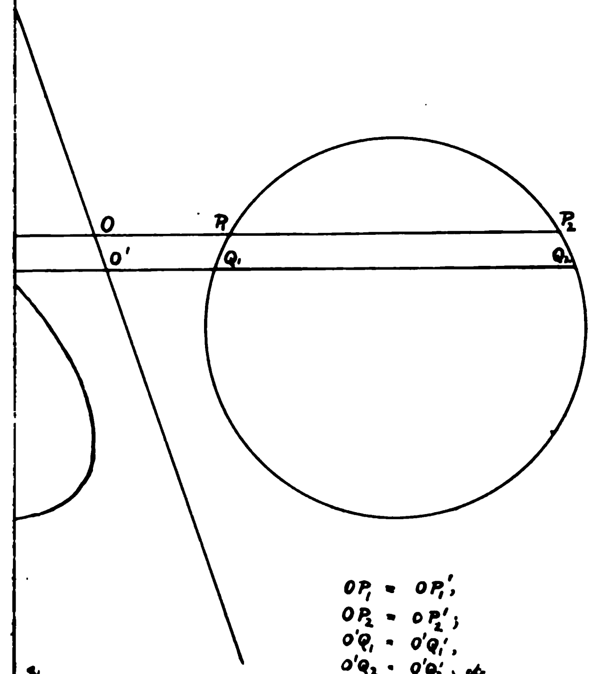

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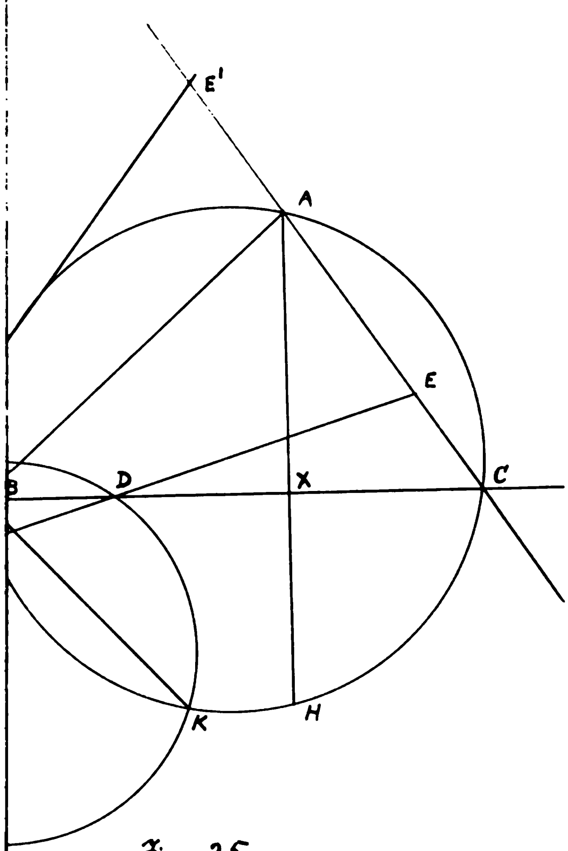

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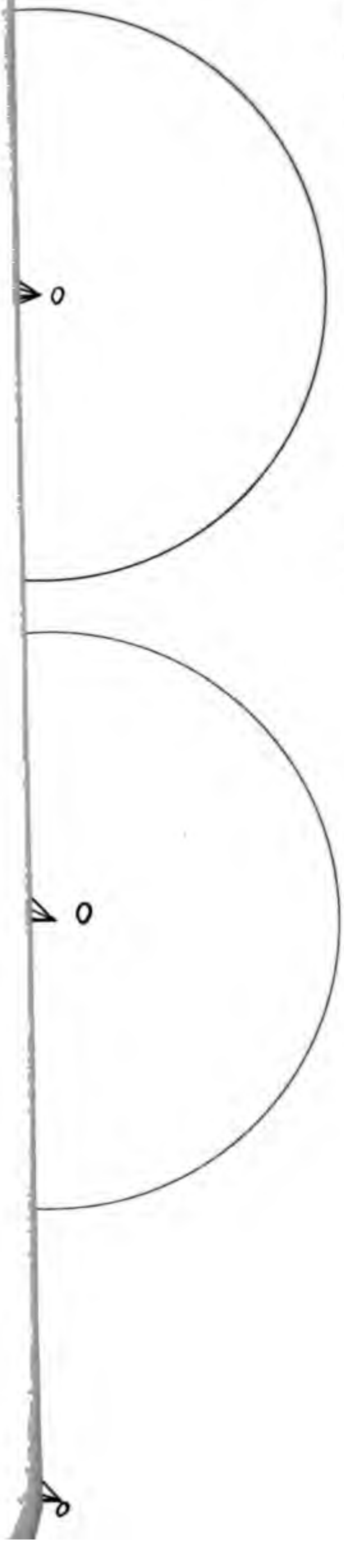
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